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Effects of shape functions on flexural–torsional buckling of fixed circular arches

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ABSTRACT

Because fixed arches have much higher flexural–torsional buckling resistance than pinended arches, they are used for engineering structures in many cases. However, studies on their flexural–torsional buckling behaviour have rarely been reported in the open literature hitherto. This paper investigates the elastic flexural–torsional buckling of fixed circular arches subjected to uniform compression and uniform bending because they play important roles in the design of steel arches against their flexural–torsional failure. One of the major difficulties in solving the flexural–torsional buckling problem of a fixed arch is to determine its accurate buckling shapes. The flexural–torsional buckling shapes are studied using a finite element (FE) method in association with eigenvalue analyses. It is found that the flexural–torsional buckling shape of a fixed arch becomes more complicated than the case of a straight beam-column or a shallow arch when the rise-to-span ratio increases, and so the theoretical analysis requires more terms of Fourier trigonometric series to describe the buckling shapes. Based on this, analytical solutions for flexural–torsional buckling loads of fixed arches are derived both by the Rayleigh–Ritz method and by solving differential equations for buckling deformations. Comparisons with the FE results show that the analytical solutions by the Rayleigh–Ritz method are reasonably accurate and that the analytical solutions by solving the equations for buckling deformations are exactly the same as the FE results. Simple approximate formulas for buckling loads of fixed arches with box-sections are proposed based on the extensive FE results for structural designers to use. The validity of the effective length method for the fixed arches is also discussed. It is found that in the case of circular arches the effective length method should not be used because the rise-to-span ratios and boundary conditions have complicated and significant influence on the buckling load.

1. Introduction

An arch under in-plane loading may suddenly displace laterally and twist out of its plane of loading and fail in a flexural–torsional buckling mode, before it buckles in the plane of loading in a bifurcation mode or in a limit point instability mode [1–8], when it does not have adequate lateral-bracings. Extensive studies on elastic flexural–torsional buckling of circular arches have been carried out by a number of researchers analytically (either by static equilibrium approaches or by energy approaches) [7–33], numerically [34–38], and experimentally [39,40]. In particular, investigations of the elastic flexural–torsional buckling load of arches in uniform compression or in uniform bending are most extensive [7–28]. Although arches may have different boundary conditions, most studies have been focused on pin-ended circular arches [7–24]. It was concluded [12–14,18,19] that it is important to use the correct strain expressions in deriving solutions for the flexural–torsional buckling load of circular arches, and that the solutions of Vlasov [8] and Yoo [10] for circular arches under uniform compression are not reliable because incorrect strains were used. Although expressions of solutions for flexural–torsional buckling loads of pin-ended arches in uniform compression or in uniform bending obtained by researchers are slightly different from each other, their numerical results are close to each other [7–24]. It has been shown [7–24,31–33] that as the included angle of a pin-ended circular arch increases, its elastic flexural–torsional buckling resistance to uniform compression and to uniform bending decreases rapidly and becomes very small for a deep arch. Therefore, the pinned-supports are undesirable for the flexural–torsional buckling resistance of deep arches. It is known that the elastic flexural–torsional buckling resistance of a fixed arch is much higher than that of its pin-ended counterpart [25–28,45] thus the arches with fixed supports are used in many cases, particularly for deep arches. Hence, investigations of elastic flexural–torsional buckling of fixed arches in uniform compression and in uniform bending are much

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Nomenclature

A	cross-sectional area of sections
b	flange width of cross-section
E	modulus of elasticity
EI_w	warping rigidity of cross-section
EI_y	flexural rigidity of cross-section with respect to out-of-plane bending
f	rise of an arch
G	shear modulus of elasticity
GJ	torsional rigidity of cross-section
h	overall height of cross-section
I_y	out-of-plane second moment of area of the cross-section
i_y	$=\sqrt{I_y/A}$, gyration radius of the cross-section with respect to out-of-plane bending
K	sectional warping parameter
k	sectional torsional parameter
L	span of an arch
M	bending moment of cross-section
M_{cr}	buckling moment of a pin-ended arch
M_{crf}	flexural–torsional buckling moment of a fixed arch in uniform bending
M_{crf}^{+}	flexural–torsional buckling moment of a fixed arch in uniform positive bending
M_{yzf}	the first mode flexural–torsional buckling moment of a uniformly bending fixed beam
m_a	buckling coefficient corresponding to the buckling load
N	axial compressive force of cross-section
N_{crf}	flexural–torsional buckling axial force of an fixed arch in uniform compression
P_{yf}	the first mode flexural buckling load of an axially-loaded fixed column
q	conservative radial distributed load
q_{crf}	radial distributed load corresponding to flexural–torsional buckling of an arch
R	radius of an arch
r_0	polar gyration radius of cross-section
S, S_f	developed length of cross-sectional centroid axis of an arch
t_f	flange thickness of cross-section
t_w	web thickness of cross-section
u	out-of-plane translational displacement of cross-sectional centroid of an arch
y_q	loading position of external force over cross-section along the y axis
Θ	subtended angle of an arch
θ	twist rotation of cross-section of an arch
λ_y	$= S/i_y$, out-of-plane slenderness of an arch
φ	coordinate of central angle of an arch

needed because they provide the essential parameters in the design of fixed arches against their flexural–torsional failure [41–46]. However, studies on the elastic flexural–torsional buckling of fixed arches in uniform compression and in uniform bending are not adequate although Timoshenko and Gere [7] had a brief discussion about their flexural–torsional buckling loads and Pi and Bradford [25,26] derived the finite strains and energy equations, and derived the buckling loads using the Rayleigh–Ritz method. Exact analytical solutions for flexural–torsional buckling loads of fixed arches in uniform compression or in uniform bending do not appear to be reported in the open literature hitherto. In addition, because exact solutions are usually complicated, approximate analytical solutions are very much desired, which are often derived by the Rayleigh–Ritz method. When the Rayleigh–Ritz method is used in association with the total potential energy of the arch and load system to derive the approximate solutions, the lateral displacement $u(\varphi)$ and twist angle $\theta(\varphi)$ during flexural–torsional buckling are usually expressed as Fourier trigonometric series [19,21,23]. And numerical methods often need to be used because they are coupled with each other in the potential energy expression. In many cases, however, when proper and adequate terms of Fourier trigonometric series for $u(\varphi)$ and $\theta(\varphi)$, which satisfy the boundary conditions, are chosen, accurate approximate analytical solutions can be derived [19,21,23]. It has been shown [12,13,19,22,23] that the single term of the trigonometric series given by

$$\frac{u}{u_0}=\frac{\theta}{\theta_0}=\cos\frac{\pi\varphi}{\Theta} \quad \text{with} \quad \varphi\in\left[-\frac{\Theta}{2},\frac{\Theta}{2}\right] \tag{1}$$

can accurately describe the first mode flexural–torsional buckling shape of pin-ended circular arches in uniform compression and in uniform bending and lead to sufficiently accurate solutions for their lowest lateral–torsional buckling loads, where u_0 and θ_0 denote the maximum lateral displacement and twist angle at the arch crown, respectively, while φ is the coordinate of central angle and Θ is the subtended angle of the arch. It has also been shown in [14,19,21,31] that the single term of the trigonometric series given by

$$\frac{u}{u_0}=\frac{\theta}{\theta_0}=\frac{1}{2}\left(1+\cos\frac{2\pi x}{S}\right) \quad \text{with} \quad x\in\left[-\frac{S}{2},\frac{S}{2}\right] \tag{2}$$

can describe the first mode flexural–torsional buckling shape of fixed beams in uniform bending or the first mode flexural buckling shape of fixed columns in uniform compression, where S is the length of the beam or column. Because of these, it may be thought that the single term similar to Eq. (2) can be used to derive the flexural–torsional buckling loads of fixed arches in uniform compression or in uniform bending. However, test results for flexural–torsional buckling of fixed arches reported in [40] have shown that the buckling shapes of the lateral displacements and twist angle during buckling are not a single sine or cosine wave because the lateral displacements $u(\varphi)$ and twist angle $\theta(\varphi)$ near the arch ends are in the opposite direction to those near the arch crown. Because the single term approximation similar to Eq. (2) cannot describe buckling shapes of $u(\varphi)$ and $\theta(\varphi)$ in the opposite directions, it may lead to incorrect prediction for the flexural–torsional loads of fixed circular arches. Therefore, to derive accurate approximate analytical solutions for the elastic flexural–torsional buckling load of fixed circular arches in uniform compression or in uniform

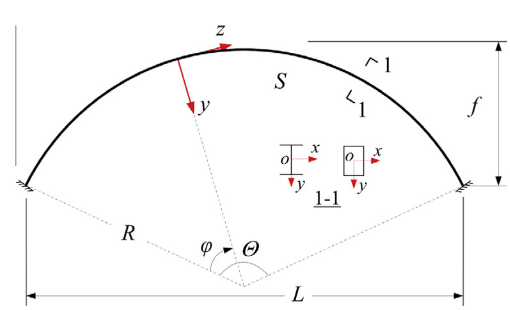

Fig. 1. Dimensions of circular arches.