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Reprint of: The best of two worlds: The expedite boundary element method

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ABSTRACT

The present developments result from the combination of the variationally-based, hybrid boundary element method and a consistent formulation of the conventional, collocation boundary element method. The procedure is simple to implement and turns out to be computationally faster than the mentioned, preceding numerical methods – and almost as accurate – for the analysis of large-scale, two-dimensional and three-dimensional problems of potential and elasticity of general shape and topology, also applicable to time-dependent problems. Both the double-layer and the single-layer potential matrices of the collocation boundary element method, **H** and **G**, respectively, whose standard evaluation requires dealing with singular and improper integrals, are obtained in an expedite way that circumvents almost any numerical integration – except for a few regular integrals. Since the resultant matrices do not differ in nature from the ones of the conventional, collocation boundary element method, the developments are suited for a matrix solution in terms of a GMRES algorithm, for example, and in the framework of the fast multi-pole method, so that very large problems can be ultimately dealt with efficiently. A few numerical examples are shown to assess the applicability of the method, its computational effort and some convergence issues.

1. Introduction

The present review paper [13] is dedicated to Prof. Herbert Mang, who visited PUC-Rio in the year 1985 to attend an IUTAM conference and subsequently delivered several lectures. One of the lectures [32] criticized the conceptual faults found in the technical literature related to the symmetrization of a stiffness-type matrix obtained in the frame of the conventional, collocation boundary element method (see e.g. [49]). With his lecture, Prof. Mang strongly influenced the first author of the present paper, who has become ever since an enthusiastic researcher of all possible, although consistent, variational unfoldings of the boundary element methods. This paper's title freely alludes to the title of Ref. [50], which has also not escaped Prof. Mang's scrutiny.

The collocation boundary element method (CBEM), whenever applicable, is a simple, powerful numerical analysis tool [4]. The present contribution is an attempt to show that the CBEM can be still more efficient and powerful – and still easier to implement computationally. (A not lesser contribution is the demonstration that simplicity can be achieved without resorting to unorthodox concepts such as the allocation of nodes away from corner points or *regularizations*.)

Some precursory works have already been published on the subject [12] or are being prepared. However, this is the first attempt to summarize the basic concepts that lead to the *expedite* boundary element method (EBEM) and to show its main features and possibilities of application in an outline that is meant to be itself expedite. The developments in Section 4 are the sequel of Refs. [14–16] and introduce some theoretical adjustments as well as some new examples.

1.1. A brief historical review

The hybrid boundary element method (HBEM) was proposed in the year 1987 [6,7] as a generalization of the concepts developed by Pian for the finite element method [41] on the basis of the Hellinger–Reissner potential [45]. It is a two-field formulation that requires only boundary integrals, since the fundamental solutions of the CBEM are used as interpolation functions. In the method, a domain of arbitrary shape and number of degrees of freedom is treated as a single finite macro-element. The price for this elegant and powerful formulation is the need of assembling a flexibility matrix and carrying out some time-consuming linear-algebra manipulations, as the resultant stiffness matrix of the problem has the format $\mathbf{K} = \mathbf{H}^{\mathrm{T}}\mathbf{F}^{*(-1)}\mathbf{H}$, where $\mathbf{F}^{*}$ is a singular (for finite domains) flexibility matrix and $\mathbf{H}$ is a kinematic transformation matrix. The latter matrix turns out to be the same double-layer potential matrix of the CBEM and it is worth observing that the adjective "hybrid", as proposed by Pian in 1967[1] also applies to

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[1] "... to signify elements which maintain either equilibrium or compatibility in the element and then to satisfy compatibility or equilibrium respectively along the interelement boundary" [42].

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mean that the present formulation combines features of both the displacement finite element [37] and the collocation boundary element methods. Although the method demands intensive use of linear algebra, advantages such as applicability, ease of post-processing results (as no further integral statements need to be resorted to) and possibility of simplifications for some particular cases make the HBEM computationally competitive. The method has been successfully applied to a variety of elasticity and potential problems including time-dependent problems, functionally graded materials and fracture mechanics [19,20,22,23,26,28,34].[2]

A simplified formulation of the HBEM has been also introduced, in which the computationally intensive evaluation of the flexibility matrix $\mathbf{F}^*$ is circumvented, as it has been shown from variational principles that $\mathbf{F}^* \approx \mathbf{HU}^*$, where $\mathbf{U}^*$ is the displacement fundamental solution measured at boundary nodes, according to Eq. (12) [5,18,20,39]. The present *expedite* boundary element method (EBEM) is a further simplification, since it comes out that whereas $\mathbf{U}^*$ may also substitute for the single-layer potential matrix $\mathbf{G}$ of the CBEM, the double-layer potential matrix $\mathbf{H}$ can be approximately evaluated without resorting to almost any integrations – compare Eqs. (10) and (21) in the subsequent developments. Then, the proposed expedite formulation – which is the core of the present paper – may lead to exactly the same matrix system of the collocation boundary element method – with the remarkable differences that the constituent matrices are evaluated in a very simple way and that post-processing of results is extremely fast. A not less remarkable difference is conceptual, since the EBEM can be outlined and implemented without the need of resorting to mathematically demanding boundary integral statements.

2. Problem formulation

An elastic body is submitted to body forces b_i in the domain Ω and traction forces $\bar{t}_i$ on part Γ_σ of the boundary. Displacements $\bar{u}_i$ are known on the complementary part Γ_u of Γ. The task is to find an adequate approximation of the stress field that satisfies equilibrium in the domain,

$$\sigma_{ji,j} + b_i = 0 \text{ in } \Omega \tag{1}$$

also satisfying the boundary equilibrium and compatibility equations,

$$\sigma_{ji} n_j = \bar{t}_i \text{ along } \Gamma_\sigma, \quad u_i = \bar{u}_i \text{ on } \Gamma_u \tag{2}$$

where n_j is the outward unit normal to Γ. The indices i, j may assume values 1, 2 or 3, as they refer to coordinate directions x, y or z, respectively, for a general 3D analysis. Summation is indicated by repeated indices. Particularization to a 2D analysis as well as to potential problems is straightforward.

2.1. Stress and displacement assumptions

Three independent fields are used in the following developments. The displacement field is explicitly approximated along the boundary by u_i^d, where $()^d$ means *displacement assumption*, in terms of polynomial functions u_{im} with compact support and nodal displacement parameters $\mathbf{d} = [d_m] \in \mathbb{R}^{n^d}$, for n^d displacement degrees of freedom of the discretized model:

$$u_i^d = u_{im} d_m \text{ on } \Gamma \tag{3}$$

such that

$$u_i^d = \bar{u}_i \text{ on } \Gamma_u \tag{4}$$

An independent stress field σ_{ij}^s, where $()^s$ stands for *stress assumption*, is given in the domain in terms of some particular solution σ_{ij}^p plus a series of fundamental solutions σ_{ijm}^* with global support, multiplied by force parameters $\mathbf{p}^* = [p_m^*] \in \mathbb{R}^{n^*}$ applied at the same boundary nodal points m to which the nodal displacements d_m are attached ($n^* = n^d$):

$$\sigma_{ij}^s = \sigma_{ijm}^* p_m^* + \sigma_{ij}^p \tag{5}$$

such that

$$\sigma_{jim,j}^* = 0 \text{ and } \sigma_{ji,j}^p = b_i \text{ in } \Omega \tag{6}$$

Displacements u_i^s are obtained from σ_{ij}^s:

$$u_i^s = u_{im}^* p_m^* + u_i^p + u_{is}^r C_{sm} p_m^* \text{ in } \Omega \tag{7}$$

where u_{im}^* are displacement fundamental solutions corresponding to σ_{ijm}^*. Rigid body motion is included in terms of functions u_{is}^r multiplied by in principle arbitrary constants $C_{sm} \in R^{n^r \times n^*}$, where n^r is the number of rigid body displacements (r.b.d.) of the discretized problem [10]. The fundamental solutions σ_{ijm}^* are used as weight functions in the CBEM. In the variational BEMs and in the EBEM, in particular, they represent domain interpolation, or trial, functions.

The third independent field is used to approximate traction forces along the boundary by t_i^t, where $()^t$ means *traction assumption*, as required in the conventional boundary element method, given as

$$t_i^t = \frac{|J|_{(at\ \ell)}}{|J|} u_{i\ell} t_\ell \equiv t_{i\ell} t_\ell \tag{8}$$

where $u_{i\ell}$ are polynomial interpolation functions with compact support and $\mathbf{t} = [t_\ell] \in \mathbb{R}^{n^t}$ are traction-force parameters. The index i refers to the coordinate directions whereas the index ℓrefers to any of the n^t traction-force degrees of freedom of the problem (thus denoting both location and orientation), for nodes adequately distributed along boundary segments of Γ. The interpolation functions $u_{i\ell}$ have the same properties of u_{in}, as presented in Eq. (3). Eq. (8) holds as $\bar{t}_i^t = t_{i\ell} \bar{t}_\ell$ along Γ_σ, in particular, according to Eq. (2).

In the above equation, $|J|_{(at\ \ell)}$ is the value of the Jacobian of the global (x,y,z) to natural (ξ,η) coordinate transformation at the nodal point ℓ; the expression $|J|_{(at\ \ell)}/|J|$ features a term in the denominator that cancels the Jacobian term of the infinitesimal boundary segment $d\Gamma = |J| d\xi d\eta$ in the integral expressions to be introduced in Eqs. (11) and (17). This not only improves the capacity of t_i^t to represent the traction forces along curved boundary segments but also simplifies the numerical integration of the related terms [11].

The numbers of degrees of freedom for traction forces n^t and displacements n^d are not necessarily the same, since more than one traction-force parameter are needed to represent tractions that are not single valued at the boundary surface, generally at nodes where adjacent boundary segments present different outward normals [11]. Then, it results that $n^t \geqslant n^d$, as t_ℓ in Eq. (8) are *traction-force attributes at the extremities of boundary segments*, whereas u_{in} in Eq. (3) are *displacement attributes at nodal points*. When consistently obtained, some matrices turn out to be rectangular, as in Eqs. (11) and (17) of the CBEM as well as in Eq. (13).

[2] It is worth remarking that, motivated by the first developments of the HBEM, Brebbia and Figueiredo [27] managed to propose an alternative formulation, which they called the *hybrid displacement boundary element method* – HDBEM. This method is based on the three-field potential formulation proposed by [30]. Although a thorough conceptual outline of the imbrications of the CBEM, the HBEM and HDBEM is still lacking, some basic investigation has already been performed by the first author of the present paper and collaborators [10,25,38]. The HBEM also conceptually resembles the Trefftz method [9,44,48].