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On the characterisation of polar fibrous composites when fibres resist bending

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ABSTRACT

This study aims to initiate research for the invention of methods appropriate for characterisation of fibre-reinforced materials that exhibit polar material behaviour due to fibre bending resistance. It thus focuses interest in the small strain regime, where there are examples of particular deformations for which non-polar linear elasticity fails to distinguish clearly the nature of a fibrous composite or even to account for the presence of fibres. Particular attention is accordingly given to the solution of the polar material version of the pure bending problem of transverse isotropic or special orthotropic plates with embedded fibres resistant in bending. It is seen that pure bending deformation enables polar fibre-reinforced materials to generate constant couple stress-field which, in turn, endorses uniqueness of the solution of the corresponding boundary value problem. In this context, by appropriately extending the validity of Clapeyron's theorem within the regime of polar linear elasticity for fibre-reinforced materials, it is shown that the solution of well-posed linear elasticity boundary value problems that generate a constant couple-stress field is unique. The well-known uniqueness of solution of conventional, non-polar linear elasticity boundary value problems is, in fact, a particular case in which the generated constant value of the couple-stress field is zero.

1. Introduction

When pioneered the non-linear theory of fibre-reinforced materials, Rivlin and his, then, student Adkins assumed deliberately, perhaps for simplicity, that fibres embedded in an isotropic material behave like perfectly flexible cords (Adkins 1951, chps. VIII and X; Adkins and Rivlin, 1955). Their consciousness in making this assumption, which underpinned relevant theoretical developments for long time afterwards (e.g., Pipkin and Rogers, 1971; Spencer, 1972, 1984), becomes clear at the beginning of the Introduction of Rivlin (1955).

This assumption is a good approximation in many cases of interest but not applicable when fibres resist bending. In their endeavour to fill in the implied theoretical gap and, hence, complete the earlier theoretical framework, Spencer and Soldatos (2007) developed a non-linear, second gradient, polar hyperelasticity theory that accounts for the bending stiffness of a single family of unidirectional fibres embedded in a relevant composite solid (see also Spencer, 2009). Moreover, Spencer and Soldatos (2007) presented also a corresponding linearized version of that theory, which is accordingly perceived as a polar material completion of conventional linear transverse isotropic elasticity; at least as far as transverse isotropy is due to presence of a single family of unidirectional fibres that resist bending.

It is recalled in this connection that linear anisotropic elasticity has been used extensively in modelling the behaviour of fibre-reinforced solids and structures subjected to small strain after the first half of the 20th century. Nevertheless, interest in and development of linear anisotropic elasticity started much earlier, in the first half of the 19th century (e.g., Love, 1944; Sokolnikoff, 1983). These facts and observations cannot easily dispute a thought that the aforementioned "perfectly flexible fibres" concept pre-existed the early non-linear elasticity models presented by Rivlin and Adkins; see also (Adkins, 1956).

Most recently, Soldatos (2014, 2015) revisited the polar material version of the linearized elasticity theory presented in Spencer and Soldatos (2007) and extended it in a manner that accounts for material anisotropy that (*i*) is due to presence of two families of unidirectional fibres that resist bending, and (*ii*) may be as advanced as its counterpart met in non-polar locally monoclinic materials. In cases that fibres are perfectly flexible, this new linear anisotropic elasticity development (Soldatos, 2014, 2015) reduces naturally to its conventional counterpart (e.g., Love, 1944; Sokolnikoff, 1983; Ting, 1996; Jones, 1998). However, when fibre bending stiffness is

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accounted for, the theory faces new challenges which are mainly due to the fact that (*i*) its governing equations are not anymore elliptic, and (*ii*) it involves a number of additional elastic moduli whose role is as yet unclear.

In conventional non-polar elasticity, loss of ellipticity of the governing equations may be observed only in the large, non-linear deformation regime, where it is associated with formation of several different modes of possible material instability. Those instability modes manifest themselves as multiple solutions of the relevant non-linear governing equations. That "loss of ellipticity" concept was first investigated by Knowles and Sternberg (1975) within the framework of non-polar isotropic hyperelasticity, while corresponding fibre-reinforced material studies emerged in Triantafyllidis and Abeyarante (1983). Research in this subject is still advancing (e.g., Dorfmann et al., 2010; El Hamdaoui and Merodio 2015), but the implied loss of ellipticity concept is already considered the source of different failure modes met in fibre-reinforced material mechanics, including fibre-kinking, fibre debonding and fibre splitting (e.g., Merodio and Ogden, 2002; 2003).

In contrast, non-polar linear elasticity guarantees the elliptic nature of its governing equations and, therefore, the uniqueness of the solution of corresponding well-posed boundary value problems; extensive relevant references as well as additional remarks and accounts maybe found, for instance, in Love (1944) and Sokolnikoff (1983). In this connection, the fact that the governing equations of the polar linear elasticity presented in Soldatos (2014,2015) are not elliptic raises, natural questions and doubts regarding potential uniqueness of the solution of corresponding polar material boundary value problems. Nevertheless, some progress in this direction is still possible, and this is reported in the present study.

On the other hand, the specific role and significance of each of the elastic moduli met in non-polar isotropic and/or anisotropic linear elasticity has been made clear long ago. This has been achieved by suitable exploitation of the solution of certain fundamental non-polar elasticity boundary value problems, such as the unidirectional compression/extension of rods, multi-directional compression/extension, and/or shear loadings of plates, etc. (e.g., Love, 1944; Sokolnikoff, 1983; Jones, 1998). However, the role of the additional elastic moduli entering the polar linear elasticity theory presented in Soldatos (2014, 2015) needs still to be investigated and become better understood. An initial effort in that direction is accordingly also presented in this paper, by revisiting and studying in detail the pure bending problem of a fibre-reinforced plate in the light of polar linear elasticity equations stemming from Soldatos (2014, 2015).

Under these considerations, Section 2 briefs the basic theoretical background required for this study, including equilibrium equations, basic kinematic and relevant constitutive equations of polar linear anisotropic elasticity. Section 3 introduces next the concepts of the displacement-gradient, the rotation and the spin energy functions. Appropriate use of these concepts enable afterwards Section 4 to develop an interesting polar material extension of Clapeyron's Theorem and, hence, to further prove uniqueness of the solution of a well-posed relevant boundary value problem when the couple-stress tensor is constant.

The latter condition guarantees uniqueness of the solution and is used afterwards in the applications considered in Sections 5 and 6. These present a number of useful solutions to different versions of the problem of pure, plane strain bending of an infinitely long fibre-reinforced plate made of perfectly flexible fibres and fibres resistant in bending, respectively. Either of Sections 5 and 6 considers material constitution which is consistent with the material symmetries of both transverse isotropy and special orthotropy. Through comparison of corresponding theoretical results, an initial attempt is thus made towards clarification of the role of the fibre bending stiffnesses met in the polar material version of linear anisotropic elasticity. Basic conclusions as well as relevant thoughts for possible continuation/extension of this study are finally summarised in Section 7.

2. Basic theory and equations

Full derivation details of the equations of polar linear elasticity of fibre-reinforced materials may be found in Soldatos (2014, 2015). This Section quotes briefly only a set of equations required for the purposes of the present study. Presentation of that set of equations within the framework of a right-handed Cartesian coordinate system Ox_i begins with the introduction of a stress tenor, $\boldsymbol{\sigma}$, and a couple-stress tensor, $\boldsymbol{m}$, which are related as follows:

$$\sigma_{ij} = \sigma_{(ij)} + \sigma_{[ij]}, \sigma_{[ij]} = \frac{1}{2}\varepsilon_{kji} m_{\ell k,\ell}. \tag{2.1}$$

Here $\sigma_{(ij)}$ and $\sigma_{[ij]}$ are the components of the symmetric and the antisymmetric part of the stress-tensor, respectively, and $\boldsymbol{\varepsilon}$ is the three-dimensional alternating tensor. Subscripts are generally assumed to take values 1, 2 and 3. Moreover, the summation notation applies on repeated indices and, in the usual manner a comma between indices denotes partial differentiation.

2.1. Equilibrium and kinematic considerations

By considering for simplicity that body forces and body couples are absent, a note is initially made of the fact that (2.1b) is essentially an equation that guarantees couple-stress equilibrium in the continuum. Then stress equilibrium considerations yield

$$\sigma_{ij,i} = \sigma_{(ij),i} + \frac{1}{2}\varepsilon_{kji}\bar{m}_{\ell k,\ell i} = 0, \tag{2.2}$$

where

$$\bar{m}_{\ell k} = m_{\ell k} - \frac{1}{3} m_{rr}\delta_{\ell k} \tag{2.3}$$

is the deviatoric part of the couple-stress tensor, and the appearing Kronecker's delta represents the components of the unit matrix, $\boldsymbol{I}$. Moreover, the components of the traction and the couple-traction vectors acting on any surface S of the fibrous composite are respectively given as follows:

$$T_i^{(n)} = \sigma_{ji} n_j, L_i^{(n)} = \bar{m}_{ji} n_j. \tag{2.4}$$

where $\boldsymbol{n}$ denotes the outward unit normal of S.

It is pointed out with interest that when polar material behaviour manifests itself through generation of a couple-stress field that is at most linear in the spatial co-ordinates, the stress equilibrium Eqs. (2.2) retain the form of their non-polar material counterpart. Moreover, in the particular case that polar material manifestation is associated with generation of a constant couple-stress field, then the antisymmetric part of stress (2.1b) is zero and, as a result, the stress field is still symmetric.

In accordance with conventional, non-polar elasticity kinematics, consider next the displacement vector $\boldsymbol{u}$ and, in the usual manner, define the linear elasticity strain and rotation tensors,

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \omega_{ij} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right). \tag{2.5}$$

as the symmetric and the antisymmetric part of the displacement gradient, respectively. Also, recall that the rotation tensor and the spin vector, $\boldsymbol{\Omega}$, are related as follows:

$$\Omega_{\mathrm{i}} = \frac{1}{2}\varepsilon_{ijk}\omega_{kj}, \omega_{kj} = \varepsilon_{ijk}\Omega_i. \tag{2.6}$$

Assume next that the material of interest contains at most two families of embedded fibres and denote with $\boldsymbol{a}^{(n)}$ the unit vector