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# A commutative-symmetrical multiplicative decomposition of left and right stretch tensors

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ABSTRACT

Symmetric positive definite left or right stretch tensors are decomposed multiplicatively into Eulerian or Lagrangean triple tensor products of symmetrizing rotations in the middle between two symmetric positive definite partial stretches. The proper orthogonal rotation tensors in the middle are determined from the symmetry conditions of the whole triple tensor products. The substitutions of the symmetrizing rotation tensors yield two commutative-symmetrical partial-stretch tensor products, which are (isotropic tensor) functions of the partial-stretch tensors of either proper Eulerian type defined with respect to a present configuration or proper Lagrangean type defined with respect to a reference configuration and which are equal to the symmetric total stretch tensors. Commutative-symmetrical partial-stretch tensor products do not rely on intermediate (stress-free) configurations. The eigenbase vector orientations of their proper Eulerian or proper Lagrangean multiplicative-elastic stretch tensors are well-defined, which is essential in order to model constitutive equations properly. Finite material orthotropy can be modeled simultaneously for both constituents and without the interference of their deformation-induced anisotropies when the partial-stretch tensors of the Lagrangean commutative-symmetrical products are defined with respect to the same reference configuration of orthotropy. The commutative-symmetrical partial-stretch tensor products are applicable to the constitutive modeling of finite anisotropy, and they constitute a novel approach to the kinematics of multiplicatively coupled total and partial stretch tensors.



## 1. Introduction

The multiplicative split of the deformation gradient F according to Bilby et al. (1957), Kröner (1960), Lee and Liu (1967), Lee (1969) and others into (say) the elastic and plastic deformation factors $\{{}^e\bar{F},\ {}^p\bar{F}\}$ is linked to the notion of intermediate configurations which constrain the elastic deformation factors ${}^e\bar{F}$ to stress-free states and which differ from the present configuration $\kappa$ or the reference configuration $\kappa_0$ of properly defined tensors of Eulerian or Lagrangean type. For non-symmetric plastic flow rules, the plastic deformation factor ${}^p\bar{F}$ is defined by the evolution equation

$$\dot{{}^p\bar{F}} = {}^p\bar{\ell}.{}^p\bar{F} \tag{1}$$

with nine internal degrees of freedom (given by the nine components ${}^p\bar{\ell}_{ij}$ of the non-symmetric plastic flow tensor ${}^p\bar{\ell}$). When the deformation gradient F and its plastic deformation factor ${}^p\bar{F}$ are known, the multiplicative Bilby–Kröner–Lee form

$$F = {}^e\bar{F}.{}^p\bar{F} \tag{2}$$

unambiguously defines the elastic deformation by

$${}^e\bar{F} = F.{}^p\bar{F}^{-1}. \tag{3}$$

The question arises: why ${}^e\bar{F}.{}^p\bar{F} \neq {}^p\bar{F}.{}^e\bar{F}$ and not ${}^p\bar{F}.{}^e\bar{F}$, which would result in another (rotated) definition of the elastic deformation (factor tensor)? From the elastic deformation definition (3), the symmetric proper Eulerian left multiplicative-elastic deformation tensor

$${}^e\bar{b} = {}^e\bar{F}.{}^e\bar{F}^T = F.{}^pC^{-1}.F^T = R.\underbrace{(U.{}^pC^{-1}.U)}_{{}^eC}.R^T \tag{4}$$

and the symmetric pseudo Lagrangean right multiplicative-elastic deformation tensor

$${}^e\bar{C} = {}^e\bar{F}^T.{}^e\bar{F} = {}^p\bar{F}^{-T}.C.{}^p\bar{F}^{-1} = {}^p\bar{R}.\underbrace{({}^pU^{-1}.C.{}^pU^{-1})}_{\widehat{{}^eC}}.{}^p\bar{R}^T \tag{5}$$

follow straightforwardly with ${}^e\bar{b} = {}^eb = ({}^ev)^2$, ${}^eC = ({}^eU)^2$ and ${}^pC = ({}^pU)^2$. The "*bar*" shall distinguish deformation tensors of multiplicative formulations for non-symmetric plastic flow rules (2) from commutative-symmetrical partial-stretch product tensors of Green and Naghdi (1965) formulations for symmetric plastic

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flow rules

$$^{e}\mathrm{C} = \mathrm{U}.{}^{p}\mathrm{C}^{-1}.\mathrm{U} \qquad \text{and} \qquad {}^{e}\mathrm{b} = \mathrm{F}.{}^{p}\mathrm{C}^{-1}.\mathrm{F}^{T} = \mathrm{R}.{}^{e}\mathrm{C}.\mathrm{R}^{T} \quad (6)$$

and also from the abbreviation

$$\widehat{^{e}\mathrm{C}} = {}^{p}\mathrm{U}^{-1}.\mathrm{C}.{}^{p}\mathrm{U}^{-1}\,. \quad (7)$$

Within a Green and Naghdi (1965) formulation for symmetric plastic flow rules

$$^{p}\dot{\mathrm{C}} = {}^{p}\mathrm{A} = {}^{p}\mathrm{A}^{T} \quad (8)$$

with six internal degrees of freedom (given by the six components ${}^{p}A_{ij} = {}^{p}A_{ji}$ of the plastic flow tensor ${}^{p}\mathrm{A}$), only the plastic right stretch tensor ${}^{p}\mathrm{U} = \sqrt{{}^{p}\mathrm{C}}$ is well defined from the flow rule's materially convected time integral but the plastic deformation factor ${}^{p}\tilde{\mathrm{F}} = {}^{p}\tilde{\mathrm{R}}.{}^{p}\mathrm{U}$ is not—because its plastic rotation tensor ${}^{p}\tilde{\mathrm{R}}$ (with three internal degrees of freedom) remains undetermined, cf. Casey and Naghdi (1980). Different authors overcome this rotational indetermination through different arbitrary assumptions on internal rotations.

In this work, a novel *commutative-symmetrical partial-stretch tensor product* is suggested which determines the three rotational degrees of freedom of a triple tensor product (consisting of a symmetrizing rotation in the middle between two partial stretches) from the symmetry of the whole product; the corresponding symmetric multiplicative-elastic deformation tensors ${}^{e}\mathrm{b}$ and ${}^{e}\mathrm{C}$ as functions of total and plastic stretches are given by (6). The tensors ${}^{e}\mathrm{b}$ and ${}^{e}\mathrm{C}$ are compared to (4) and (5) or to multiplicative-elastic deformation tensors of the Lee (1969) or the Drucker (1985) multiplicative model with arbitrary rotational constraints. It turns out that only the right multiplicative-elastic deformation tensors of the four multiplicative models differ with respect to the orientation of their right eigenbase vectors; their eigenvalues and the left multiplicative-elastic deformation tensors (including the left eigenbase vectors) are identical. The multiplicative-elastic deformation tensors (6) may further be employed as tensor arguments for the definition of logarithmic strain tensors in order to generalize *additive* to *multiplicative* logarithmic strain space formulations.

## 2. Preliminaries and notation

The deformation gradient

$$\mathrm{F} = \partial\mathbf{x}/\partial\mathbf{X} = \hat{U}_k\,\underbrace{\hat{\mathbf{e}}_k \otimes \hat{\mathbf{E}}_k}_{\mathrm{R}} = \underbrace{\hat{U}_k\,\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_k}_{\mathrm{v}}.\mathrm{R} = \mathrm{R}.\underbrace{\hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k \hat{U}_k}_{\mathrm{U}}, \quad (9)$$

defined as the partial derivative $\partial\mathbf{x}/\partial\mathbf{X}$ of the vicinity $d\mathbf{x}$ of the present position vector $\mathbf{x}$ (within the present configuration $\kappa$) with respect to the vicinity $d\mathbf{X}$ of the reference position vector $\mathbf{X}$ (within the reference configuration $\kappa_0$), reveals in its spectral representation (9) the multiplicative decoupling—in line with the polar decomposition theorem (Thomson and Tait, 1879; Richter, 1952)—into a proper orthogonal (orthonormal $\mathrm{R}^{-1} = \mathrm{R}^{T}$ and right-handed $\det(\mathrm{R}) = 1$) material rotation tensor

$$\mathrm{R} = \hat{\mathbf{e}}_k \otimes \hat{\mathbf{E}}_k \quad (10)$$

and into the symmetric positive definite left or right stretch tensors

$$\mathrm{v}^{T} = \mathrm{v} = \hat{U}_k\,\hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_k \qquad \text{or} \qquad \mathrm{U}^{T} = \mathrm{U} = \hat{U}_k\,\hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k\,. \quad (11)$$

The summation convention is adopted on multiple repeated indices

$$\hat{U}_k\,\hat{\mathbf{e}}_k \otimes \hat{\mathbf{E}}_k = \hat{U}_1\,\hat{\mathbf{e}}_1 \otimes \hat{\mathbf{E}}_1 + \hat{U}_2\,\hat{\mathbf{e}}_2 \otimes \hat{\mathbf{E}}_2 + \hat{U}_3\,\hat{\mathbf{e}}_3 \otimes \hat{\mathbf{E}}_3\,. \quad (12)$$

Throughout this work, eigenvalues $\hat{U}_k$ and left $\hat{\mathbf{e}}_k$ or right $\hat{\mathbf{E}}_k$ eigenbase vectors are marked with a "*hat*". The eigenbase vectors $\hat{\mathbf{e}}_k$ and $\hat{\mathbf{E}}_k$, which are defined by the normalized left and right eigenvectors, are orthogonal and obey $\hat{\mathbf{e}}_i.\hat{\mathbf{e}}_j = \hat{\mathbf{E}}_i.\hat{\mathbf{E}}_j = \delta_{ij}$, where $\delta_{ij}$ denotes Kronecker's delta, i.e. the $\mathbf{E}_i \otimes \mathbf{E}_j$ components of the second-order identity tensor I. The transpose, the inverse, the transposed inverse and the determinant of a second-order tensor A are $\mathrm{A}^{T}$, $\mathrm{A}^{-1}$, $\mathrm{A}^{-T}$ and $\det(\mathrm{A})$, respectively. The '$\otimes$' denotes the *dyadic product operator*, and a dot '.' a *dot product operator* (or single contraction) $\mathbf{a}.\mathbf{b} = a_k b_k = \mathrm{tr}(\mathbf{a} \otimes \mathbf{b})$, defined here by the *trace* of the $\mathbf{a} \otimes \mathbf{b}$ vector dyad. The positive definiteness ($\hat{U}_k > 0$) of the deformation gradient F implies: $\det(\mathrm{F}) > 0$. The left and right stretch tensors (11) as well as the corresponding symmetric positive definite left [(Cauchy, 1827), p.62, Eqs. (10) and (11)] and right [(Green, 1839), pp.123–124] Cauchy–Green deformation tensors

$$\mathrm{b}^{T} = \mathrm{b} = \mathrm{F}.\mathrm{F}^{T} = \mathrm{v}^{2} = \left(\hat{U}_k\right)^{2} \hat{\mathbf{e}}_k \otimes \hat{\mathbf{e}}_k$$

$$\text{and} \quad \mathrm{C}^{T} = \mathrm{C} = \mathrm{F}^{T}.\mathrm{F} = \mathrm{U}^{2} = \left(\hat{U}_k\right)^{2} \hat{\mathbf{E}}_k \otimes \hat{\mathbf{E}}_k \quad (13)$$

are, respectively, the material R-forward or $\mathrm{R}^{T}$-backward rotations

$$\mathrm{b} = \mathrm{R}.\mathrm{C}.\mathrm{R}^{T}\,, \quad \mathrm{v} = \mathrm{R}.\mathrm{U}.\mathrm{R}^{T} \quad \text{or} \quad \mathrm{C} = \mathrm{R}^{T}.\mathrm{b}.\mathrm{R}\,, \quad \mathrm{U} = \mathrm{R}^{T}.\mathrm{v}.\mathrm{R} \quad (14)$$

of each other—as are, from equation (9), the Eulerian or Lagrangean eigenbase vectors

$$\hat{\mathbf{e}}_k = \mathrm{R}.\hat{\mathbf{E}}_k = \hat{\mathbf{E}}_k.\mathrm{R}^{T} \qquad \text{or} \qquad \hat{\mathbf{E}}_k = \mathrm{R}^{T}.\hat{\mathbf{e}}_k = \hat{\mathbf{e}}_k.\mathrm{R}, \quad (15)$$

i.e. the normalized and orthogonal left or right eigenvectors. The contractions within tensor products are always denoted with "*dots*" and the labels "e" or "p" of partial tensors are written as upper left indices in order to distinguish them better from transposes, transposed inverses or (isotropic tensor) power functions. Even if most of this work focuses on elasto-plasticity, its kinematic key elements remain valid for inelasticity in general (and the label "*p*" for *plastic* may also be interpreted as *inelastic*). Under *superposed rigid body motions* (tagged with a "+") *with a rotation* Q (without "*tilde*")

- the deformation gradient (9) and the material rotation tensor (10) are related by

$$\mathrm{F}_{+} = \mathrm{Q}.\mathrm{F} \qquad \text{and} \qquad \mathrm{R}_{+} = \mathrm{Q}.\mathrm{R} \quad (16)$$

- *proper Eulerian tensors* (written in lowercase) with respect to the present configuration $\kappa$ are *altered* and related by

$$\mathrm{b}_{+} = \mathrm{Q}.\mathrm{b}.\mathrm{Q}^{T}\,, \qquad \mathrm{v}_{+} = \mathrm{Q}.\mathrm{v}.\mathrm{Q}^{T} \quad (17)$$

- *proper Lagrangean tensors* (written in uppercase) with respect to the reference configuration $\kappa_0$ are *invariant*

$$\mathrm{C}_{+} = \mathrm{C}\,, \qquad \mathrm{U}_{+} = \mathrm{U} \quad (18)$$

## 3. The multiplicative Bilby–Kröner–Lee form for symmetric plastic flow rules

For symmetric plastic flow rules (8), the multiplicative Bilby–Kröner–Lee form

$$\mathrm{F} = {}^{e}\tilde{\mathrm{F}}.(\tilde{\mathrm{Q}}^{T}.\tilde{\mathrm{Q}}).{}^{p}\tilde{\mathrm{F}} = ({}^{e}\tilde{\mathrm{F}}.\tilde{\mathrm{Q}}^{T}).(\tilde{\mathrm{Q}}.{}^{p}\tilde{\mathrm{F}}) \quad (19)$$

is undetermined with respect to an internal rotation $\tilde{\mathrm{Q}}$—as are ${}^{e}\tilde{\mathrm{F}}$ and ${}^{p}\tilde{\mathrm{F}}$. Marking indeterminable tensors with "*tildes*" and applying the polar decompositions of the undetermined elastic and plastic deformation factors

$$^{e}\tilde{\mathrm{F}} = {}^{e}\mathrm{v}.{}^{e}\tilde{\mathrm{R}} = {}^{e}\tilde{\mathrm{R}}.{}^{e}\tilde{\mathrm{U}} \qquad \text{and} \qquad {}^{p}\tilde{\mathrm{F}} = {}^{p}\tilde{\mathrm{v}}.{}^{p}\tilde{\mathrm{R}} = {}^{p}\tilde{\mathrm{R}}.{}^{p}\mathrm{U} \quad (20)$$

to the multiplicative Bilby–Kröner–Lee form (19), reveals

$$^{e}\tilde{\mathrm{F}}.\underbrace{(\tilde{\mathrm{Q}}^{T}.\tilde{\mathrm{Q}})}_{\mathrm{I}}.{}^{p}\tilde{\mathrm{F}} = \underbrace{{}^{e}\mathrm{v}.{}^{e}\tilde{\mathrm{R}}}_{{}^{e}\tilde{\mathrm{R}}.{}^{e}\tilde{\mathrm{U}}}.(\tilde{\mathrm{Q}}^{T}.\tilde{\mathrm{Q}}).\underbrace{{}^{p}\tilde{\mathrm{R}}.{}^{p}\mathrm{U}}_{{}^{p}\tilde{\mathrm{v}}.{}^{p}\tilde{\mathrm{R}}} = {}^{e}\mathrm{v}.\underbrace{({}^{e}\tilde{\mathrm{R}}.\tilde{\mathrm{Q}}^{T}).(\tilde{\mathrm{Q}}.{}^{p}\tilde{\mathrm{R}})}_{\mathrm{O}}.{}^{p}\mathrm{U} = {}^{e}\mathrm{v}.\mathrm{O}.{}^{p}\mathrm{U} \quad (21)$$