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Derivation of complete crack-tip stress expansions from Westergaard–Sanford solutions

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ABSTRACT

The description of mechanical fields at the vicinity of a bi-dimensional crack-tip can be performed using the classic Williams series expansion. While its general structure is well known, complete expressions are rarely available for specific problems. This article describes and applies a methodology to express complete expansions for four given fracture configurations. The procedure relates Williams series coefficients to those of expanded Westergaard–Sanford complex potentials for modes I and II. Actual expansions of complex solutions for mode I cases are derived using classical complex analysis techniques. Complete closed-form results, four power series and one Laurent series, have been determined with this approach. The correctness of analytical results and series convergence behavior have been conclusively investigated through numerical tests comparing reference complex solutions with truncated series representations. The methodology can be applied straightforwardly to new fracture configurations where complex solutions are known. Complete closed-form expressions can be used to derive, test and improve numerical and experimental techniques involving higher order terms in crack-tip expansions.

1. Introduction

Williams asymptotic expansion (Williams, 1957) provides an analytical framework for the description of elasticity solutions in 2D solids with straight load-free wedges or cracks. In the context of linear elastic fracture mechanics this formalism represents mechanical fields as series expressed in a crack-tip local polar coordinate system (Fig. 1). Williams approach establishes the separate influence of radial and angular dependencies and can take into account the linear superposition of the three fracture modes. The stress field can then be expressed as:

$$\sigma_{ij}(r,\theta) = \sum_{m=1}^{3} \sum_{k=-\infty}^{\infty} a_k^m \cdot f_k^{m,ij}(\theta) \cdot r^{\frac{k}{2}-1} \tag{1}$$

with a_k^m being configuration-specific coefficients and $f_k^{m,ij}(\theta)$ being general angular eigenfunctions. From the whole series, terms with $k \leq 0$ are traditionally discarded so as to ensure the finiteness of elastic strain energy around the crack-tip. Among terms of order $k \geq 1$, the first one is generally the only one considered. Having a singular radial dependency $r^{-1/2}$, it is expected to be the most influential term close to the crack-tip and is often sufficient for the study of brittle fracture (Wieghardt, 1907; Irwin, 1957; Wieghardt et al., 1995). The magnitude of this term is commensurate with the value of the Stress Intensity Factor (SIF). However, for some other applications, the second term in the summation has to be taken into account too (Gupta et al., 2015). This term is constant and is proportional to the well-known T-stress value for σ_{11}. SIF and T-stress have been collected for many fracture configurations using analytical, semi-analytical, experimental and numerical methods (Sherry et al., 1995; Fett, 1998; Tada et al., 2000; Murakami, 2001). Nevertheless, these sole two terms may not provide enough information to describe accurately the solution when considering mechanical fields beyond the vicinity of the crack-tip. Higher order terms have indeed to be taken into account for the sake of precision in the analysis of experimental data, the formulation of numerical methods and the derivation of analytical models for fracture mechanics.

In the context of experimental fracture mechanics, the need for higher order terms arises for the correct post-treatment of data obtained with different techniques: Moiré interferometry (Barker et al., 1985; Rozenburg et al., 2007), photoelasticity (Sanford, 1989; Ramesh et al., 1997; París et al., 1997; Guagliano et al., 2011; Stepanova et al., 2016, 2017), caustics (Kobayashi, 1993), Digital Image Correlation (Ayatollahi and Moazzami, 2017), or from computational tools (Berto and Lazzarin, 2010; Ayatollahi and Nejati, 2011; Berto and Lazzarin, 2013; Veselý et al., 2015; Malíková and Veselý, 2015; Malíková, 2015; Akbardoost and Rastin, 2015; Veselý et al., 2016). The fitting procedures are either based on a least-square approach like the over-deterministic method or a direct interpo-

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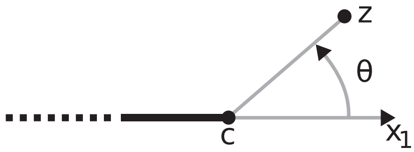

Fig. 1. Crack-tip local polar coordinate system.

lation technique. A better correlation is observed between the experimental data and the fitted series models when several higher order terms are taken into account.

Higher order terms are also considered in some numerical methods devoted the simulation of cracked solids. Their formulations include these terms so as to use a physically sound approximation basis for the unknown mechanical fields. Such numerical methods include: the Hybrid Crack Element method (Karihaloo and Xiao, 2001a,c,b; Xiao and Karihaloo, 2002, 2003; Karihaloo et al., 2003; Xiao et al., 2004; Xiao and Karihaloo, 2004; 2007a,b), the Scaled Boundary Methods (Song, 2005; Chidgzey and Deeks, 2005; Shrestha and Ohga, 2006; He et al., 2012), the Fractal Finite-Element Method (Su and Feng, 2005; Su and Fok, 2007; Treifi et al., 2008, 2009), a singular integral approach (Ananthasayanam et al., 2007), the Super Singular Element Method (Tsang and Oyadiji, 2008), the coupling of an analytical patch with a computational method (Passieux et al., 2011; Cheng et al., 2012a,b; 2013; Liao et al., 2015; Liu et al., 2015, 2016), or the finite element discretized symplectic method (Leung et al., 2014). The accuracy and convergence speed of these methods have been shown to improve when higher order terms are added to the crack-tip fields approximations.

Regarding analytical results about expressions for higher order terms, it appears that the amount of publications on the subject is relatively reduced compared to experimental and numerical applications. Among the published materials, a notable effort has concerned the justification of their role when a nonlinear zone exists in the crack-tip area. It has been shown that higher order terms are required so as to describe elastic fields outside a disk containing the nonlinear zone and to contribute to elastic strain energy inside (Hui and Ruina, 1995; Chen and Hasebe, 1997; Jeon and Im, 2001; Jeon et al., 2003; Zappalorto and Lazzarin, 2011). Closed-from expressions of series higher order terms for several fracture configurations are described in Paris (2002) and Tada and Paris (2005). However these results only describe the first few terms in expansions and concern complex solutions expressed with Westergaard approach (Westergaard, 1939). Beside classical series higher order terms, other terms with a logarithmic radial influence can also be included in the summation to account for non-linear material behavior according to (Christopher et al., 2007). And finally to the best of author's knowledge, complete closed-form series solutions have only been provided for fracture configurations in an infinite medium submitted to remote stresses at infinity (Theocaris and Spyropoulos, 1983; Yan and Yang, 1993; Hello et al., 2012; Stepanova and Roslyakov, 2016). All results are obtained from existing complex solutions expressed with the Kolosov–Muskelishvili formalism (Kolosov, 1909; Muskhelishvili, 1953). In Theocaris and Spyropoulos (1983), closed-form stress power series expressions for a slant crack under biaxial loading are derived. The authors apply these formulae to study the influence of higher order terms for the construction of photoelastic isochromatic fringe patterns. Yan and Yang (1993) described analytical solutions for a crack under uniaxial mode I and mode II solicitations. Power series solutions are given for both modes and a Laurent series expansion is provided for mode I. The accuracy of series solutions and the influence of convergence domains are tested numerically with respect to reference real solutions based on a bipolar parametrization. More recently Hello et al. (2012) obtained results related to previous ones and completed them with a thorough description of the coefficients derivation technique from complex solutions, results for mode I bi-axial loads, a Laurent series solution for mode II, an analytical demonstration of series convergence and a numerical exploration of convergence behavior with comparisons to reference complex solutions. The latest contribution by Stepanova and Roslyakov (2016) provided closed-form power series solutions for the configuration of two collinear cracks of finite lengths. Beyond their core analytical results, the authors conclusively validate and test their solutions with extensive numerical investigations concerning the influence of series truncation in mode I, mode II and mixed problems.

In this context of relative scarcity of analytical results regarding higher order terms, especially about complete closed-form series expressions, the purpose of the present article will then be to:

1. generalize the methodology described in Hello et al. (2012) for deriving closed-form power series in the context of initial complex solutions expressed with the Westergaard–Sanford formalism,
2. apply the methodology to derive new complete closed-form power series solutions for four practical fracture configurations and identify associated Williams coefficients,
3. derive a new Laurent series solution for one of the configurations,
4. validate analytical solutions and investigate their convergence behavior through numerical analyses performed with truncated series.

2. Representation of the stress field for fracture problems

2.1. Williams crack-tip stress series

Williams model (Williams, 1957) expresses the stress field as a series expansion at the crack-tip (1):

$$\sigma_{ij}(r,\theta) = \sum_{m=1}^{3} \sum_{k=-\infty}^{\infty} a_k^m \cdot f_k^{m,ij}(\theta) \cdot r^{\frac{k}{2}-1}$$

It requires the definition of an infinite set of terms to provide an exact solution. Each term of the summation is the product of three factors: a problem specific coefficient a_k^m, a general angular eigenfunction $f_k^{m,ij}(\theta)$ and a general radial power value. Analytical expressions for angular eigenfunctions are provided in the literature (Owen and Fawkes, 1983; Karihaloo and Xiao, 2001a; Kuna, 2013):

$$f_k^{1,11}(\theta) = \frac{k}{2}\begin{bmatrix} (2+k/2+(-1)^k)\cos(k/2-1)\theta \\ -(k/2-1)\cos(k/2-3)\theta \end{bmatrix} \tag{2}$$

$$f_k^{1,22}(\theta) = \frac{k}{2}\begin{bmatrix} (2-k/2-(-1)^k)\cos(k/2-1)\theta \\ +(k/2-1)\cos(k/2-3)\theta \end{bmatrix} \tag{3}$$

$$f_k^{1,12}(\theta) = \frac{k}{2}\begin{bmatrix} (k/2-1)\sin(k/2-3)\theta \\ -(k/2+(-1)^k)\sin(k/2-1)\theta \end{bmatrix} \tag{4}$$

$$f_k^{2,11}(\theta) = -\frac{k}{2}\begin{bmatrix} (2+k/2-(-1)^k)\sin(k/2-1)\theta \\ -(k/2-1)\sin(k/2-3)\theta \end{bmatrix} \tag{5}$$

$$f_k^{2,22}(\theta) = -\frac{k}{2}\begin{bmatrix} (2-k/2+(-1)^k)\sin(k/2-1)\theta \\ +(k/2-1)\sin(k/2-3)\theta \end{bmatrix} \tag{6}$$

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