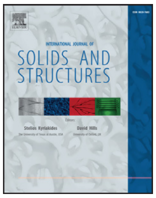
International Journal of Solids and Structures 000 (2018) 1–11

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr



Approximate semi-analytical solution for a penny-shaped rough-walled hydraulic fracture driven by turbulent fluid in an impermeable rock

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ARTICLE INFO

Article history:
Received 1 September 2017
Revised 23 February 2018
Available online xxx

Keywords:
Hydraulic fracture model
Turbulent flow
Gauckler–Manning–Strickler
Orthogonal polynomial series
Radial flow

ABSTRACT

The popularity of high injection rate hydraulic fracturing treatments using low viscosity fluids is driving a need to consider the turbulent and laminar-turbulent transition regimes of fluid flow in hydraulic fracture simulators. The radial model is one of the most important geometries both for benchmarking and as a starter solution for 3D and Planar 3D models. Here we provide a semi-analytical, orthogonal polynomial series solution for a rough-walled radial (penny-shaped) hydraulic fracture driven by a fully turbulent fluid. Embedding the appropriate pressure singularities in a family of orthogonal polynomials used for derivation of the solution leads to very rapid convergence of the series, requiring just two terms for an accurate result. We conclude with an investigation of the occurrence of this limiting regime by comparison with numerical simulations, illustrating that the fully turbulent regime is typically not encountered for the radial geometry, although the present solution remains necessary as a starter solution and benchmark for the numerical simulators that are required to capture the laminar-turbulent transition. By comparison with numerical simulations that consider the laminar-turbulent transition, we find that such an estimate is expected to be sufficient for practical purposes when the inlet opening predicted by the turbulent solution exceeds the inlet opening predicted by the laminar solution.

1. Introduction

The growing popularity of high rate fluid injection using low viscosity fluids, such as water, is one of the key characteristics of modern hydraulic fracturing (HF) King (2010). As a result, there are an increasing number of practically-relevant cases where the laminar flow assumption used in many HF models is not satisfied, at least over some non-negligible portion of the fracture. While most HF models continue to embed a laminar flow assumption (see the review of Adachi et al. (2007)), which is indeed sometimes valid, the need to consider the turbulent regime dates back at least as far as the seminal early work of Perkins and Kern (1961), who developed laminar and turbulent flow equations for a vertically-oriented blade-shaped HF, while focusing on only the laminar flow regime for radial HFs. Later contributions include Nilson (1981, 1988), which investigate the influence of turbulent flow on plane strain and radial HFs with a constant pressure inlet boundary condition. Also, Emerman et al. (1986) and Siriwardane and Layne (1991) have studied the plane-strain HF with constant inlet fluid flow for laminar and turbulent regimes.

More recently there is a growing recognition of the relevance of turbulent flow for HF growth. For example, Ames and Bunger (2015) demonstrate the potential for incorrect assumption of laminar flow to lead to poor predictions of HF length, width, and pressure. There has also been a deepening appreciation for not only the importance, but also for the subtleties and complexities of the mathematical problem and physical phenomena associated with turbulent and/or laminar-turbulent transition fluid flow in HF propagation. The complicated multi-scale structure of a turbulent HF is explored by Dontsov (2016b), who analyzed the near-tip transition from turbulent to laminar flow using the Churchill approximation to find the friction factor and the Darcy–Weisbach equation to find an asymptotic solution for a fully turbulent HF. Moreover, Zia and Lecampion (2016)); Zia and Lecampion (2017) investigate the effect of turbulent flow on height contained HF. They develop a semi-analytical solution for fully rough and smooth flow in a con-

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https://doi.org/10.1016/j.ijsolstr.2018.03.002

Please cite this article as: N. Zolfaghari, A.P. Bunger, Approximate semi-analytical solution for a penny-shaped rough-walled hydraulic fracture driven by turbulent fluid in an impermeable rock, International Journal of Solids and Structures (2018), https://doi.org/10.1016/j.ijsolstr.2018.03.002

tained (fixed-height) HF. Furthermore, they applied a drag reduction method from Yang and Dou (2010) to numerically model the transition from laminar to turbulent regimes, again for a contained HF. We note that one of the benefits of using the drag reduction method is the ability to extend the model to account for the effect of proppant and/or drag reducers, as is considered by Zia and Lecampion (2017).

Along with these recent studies focusing on a more general modeling framework for HF growth in turbulent and transition regimes, several contributions comprise an expanding family of semi-analytical solutions. These solutions are primarily useful for benchmarking numerical simulators and also for rapidly computing fracture dimensions in certain simple geometries. These include:

- Kano et al. (2015) develop an analytical solution for large leak-off PKN model using Gauckler–Manning–Strickler (GMS Gauckler, 1867; Manning, 1891; Strickler, 1981) solution for a rough-walled open channel.
- Zolfaghari et al. (in press) provide a semi-analytical solution for the blade-shaped (so-called "PKN") geometry in an impermeable rock (no leak-off) using a general form of the GMS model. This work uses a truncated polynomial series to derive a solution for fully turbulent HF, showing also the crack tip behavior and providing an alternative method to describe the transition from laminar to turbulent flow.
- Zolfaghari et al. (2017) derive a semi-analytical solution for the plane-strain geometry with no leak-off, providing an asymptotic solution for a zero-toughness plane-strain HF in the rough-walled fully turbulent regime. They also compared their result with a numerical solution that uses the Churchill approximation.

In this study, we present a semi-analytical solution for a rough-walled, fully turbulent, radial HF. We use a general form of GMS to model fluid flow within the HF. Then, following the approach taken by Savitski and Detournay (2002), we use a Jacobi polynomial series to develop the solution. The tip solution is embedded in the polynomial series to enable rapid convergence. This extension of the approaches of Savitski and Detournay (2002) and Zolfaghari et al. (2017) is non-trivial because the nature of the radial solution leads to some unique challenges. Most notably, the pressure singularity at the inlet is much stronger in the turbulent regime than in the laminar regime, with the consequence of the need to mitigate unbounded values of the crack opening at the center of the HF in the turbulent regime, whereas the opening is always finite in the laminar regime. Also the form of the pressure and opening singularity at the leading edge of the HF is also different from the laminar regime; in order to obtain rapid convergence our solution must account for this unique near-tip behavior. Finally, because the fluid flux for radial flow decays as one moves away from the inlet—in contrast to linear flow encountered in the plane strain and PKN models—the flow regime is much more prone to being in the laminar-turbulent transition at a scale that cannot be assumed small relative to the total size of the fracture. To this latter point, we clarify the applicability of the solution by way of comparison to numerical results from our companion paper, Zolfaghari and Bunger (in press), in which we develop a numerical solution to analyze the transition of turbulent flow to laminar flow in a radial HF.

As a clarification, throughout this paper we refer to "convergence" of the series. Such a trait of series solutions in hydraulic fracturing, and in particular the ability to improve convergence through embedding of appropriate tip and/or inlet behavior, is discussed in all of the prior contributions in which such methodology was established (Savitski and Detournay, 2002; Adachi, 2001; Bunger and Detournay, 2007). In this context, "convergence" refers to the stabilization of the solution to consistent values to a certain number of digits for truncation of the series after a given number of terms. Hence, while we do not formally prove convergence of the series nor do we demonstrate convergence for many terms, based on the behavior of the first several terms of the series we infer the convergence of the series.

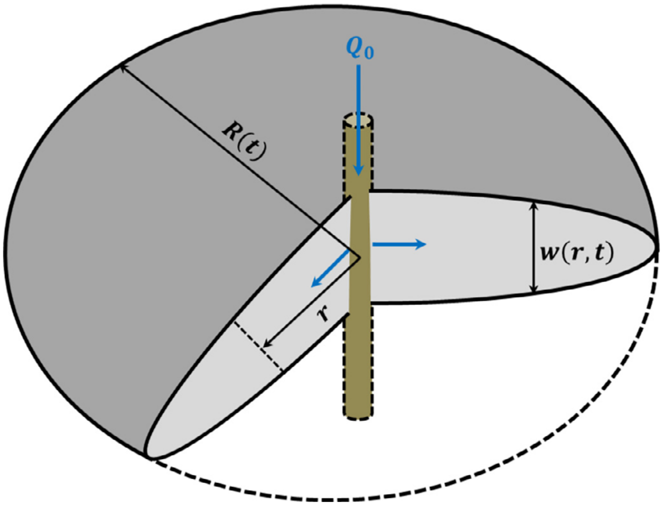

Fig. 1. Radial hydraulic fracture geometry.

The main result is that, for a radial, rough-walled HF driven by a fully turbulent fluid described by the GMS model, the width, pressure, and radius are given by

$$
\begin{aligned}
w &= \Bigg[\left(0.694+0.6148\frac{r}{R}\right)\left(1-\frac{r}{R}\right)^{\frac{6}{7}}-0.275\sqrt{1-\left(\frac{r}{R}\right)^{2}} \\
&\quad -0.6798\left(\frac{r}{R}\right)^{0.31}+0.8873\ {}_2F_1\left(\frac{1}{2},-0.155;0.845;\left(\frac{r}{R}\right)^{2}\right)\Bigg] \\
&\quad \times\left(\frac{Q_0}{\beta'\sqrt{E'}}\right)^{\frac{6}{13}} \\
p &= \left[1.0452-\frac{0.7683}{(1-r/R)^{\frac{1}{7}}}+0.0967\left(\frac{r}{R}\right)^{-0.69}\right]\left(\frac{E'^{17}Q_0^5}{\beta'^{18}}\right)^{\frac{1}{26}}t^{-\frac{1}{2}} \\
R &= 0.854\left(\beta' E'^{\frac{1}{2}}Q_0^{\frac{7}{6}}\right)^{\frac{3}{13}}t^{\frac{1}{2}}
\end{aligned}
\tag{1}
$$

where Q_0 is the fluid flow, t is time, r is the coordinate (see Fig. 1), and β' and E' are given in Eq. (5). This solution comprises the first two terms of the orthogonal polynomial series, which we demonstrate to be sufficiently accurate for most benchmarking and estimation purposes. In what follows, we will describe the mathematical model, solution method, and range of validity of this semi-analytical solution.

2. Method

The purpose of this paper is to model the effect of turbulent flow on penny shaped (radial) hydraulic fracture, where the radius of the crack is defined as $R(t)$ (see Fig. 1). In this model the radius of the wellbore is negligible with respect to crack radius, and hence the fluid is taken to be supplied from a point source at the center of the HF (Fig. 1) with constant flow rate, Q_0. Also the width, net pressure, and fluid flux at any time and at any location, r, is given by $w(r, t)$, $p(r, t)$, and $q(r, t)$, respectively, noting that net pressure $p(r, t)$ is the total fluid pressure minus the far-field stress. Considering the GMS model (Gauckler, 1867; Manning,

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