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One dimensional Willis-form equations can retain time synchronization under spatial transformations

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ABSTRACT

This paper gives a theoretical proof that the one dimensional (1D) Willis-form equations with displacement coupling terms can retain time synchronization under spatial transformations. This is also supported by a numerical example that compares the distributions of the wave velocities of an inhomogeneous material calculated by the Willis-form equations and the classical elastodynamic equations, respectively. It further demonstrates that the classical elastodynamic equations are good approximations of the Willis-form equations only when the wave frequency is sufficiently high. These findings serve as additional evidence of the validity of the Willis-form equations for inhomogeneous media.

1. Introduction

Transformation methods play important roles in the designing of metamaterials (Pendry et al., 2006; Schurig et al., 2006; Cummer and Schurig, 2007; Chen and Chan, 2010; Chen et al., 2010). These methods transform wave equations from a virtual space to a physical space of interest. The material parameters in the physical space can be obtained by comparing the wave equations before and after the transformation if they are form-invariant. Usually, the media in the virtual space are homogeneous, while the media in the physical space are inhomogeneous and are commonly called 'transformation media'. The prerequisite to using transformation methods is the form-invariant property of the wave equations under a general spatial transformation, which is true for electromagnetic (Pendry et al., 2006; Chen et al., 2010) and acoustic waves (Chen and Chan, 2010). However, Milton et al. (2006) proved that the classical elastodynamic equations in frequency domain will change to the Willis equations (Willis, 1981; Willis, 1997) after the transformation when the transformation gauge is taken as the deformation gradient (Norris and Shuvalov, 2011). This means the classical elastodynamic equations may fail to accurately describe elastic wave propagations in transformed inhomogeneous media, and therefore cannot be used to design ideal elastic metamaterials. This is the reason why people can only control bending waves in a plate when strains are of the von-Kármán form (Farhat et al., 2009) or approximately control in-plane elastic waves (Hu et al., 2011; Norris and Parnell, 2012), based on classical elastodynamic equations.

The Willis equations were originally proposed for inhomogeneous media by using the perturbation method (Willis, 1981, 1997). They contain a constitutive equation

$$\langle \sigma \rangle = C_{eff} \langle e \rangle + S_{eff} \langle \dot{u} \rangle, \tag{1a}$$

and an equation of motion

$$\nabla \cdot \langle \sigma \rangle + f = S^{\dagger}_{eff} \langle \dot{e} \rangle + \rho_{eff} \langle \ddot{u} \rangle, \tag{1b}$$

where $\boldsymbol{\sigma}$**,** $\boldsymbol{e}$**,** $\boldsymbol{u}$**,** $\boldsymbol{f}$ are stress, small strain, displacement, and body force, respectively; the angle bracket denotes ensemble average; C_{eff}, S_{eff}, $S^{\dagger}_{eff}$ (the adjoint of S_{eff}) and $\boldsymbol{\rho}_{eff}$ are non-local operators; and the overhead dot denotes time derivative. Since the structure of Willis equations does not change after a transformation (form-invariance, see Milton et al., 2006), these equations can be used to design perfect elastic wave rotators and nearly perfect elastic wave cloaks (Xiang, 2014). This gives evidence that the Willis equations outperform the classical elastodynamic equations for inhomogeneous media.

The Willis-form equations obtained by Xiang (2014) in time domain are slightly different from Eq. (1), because they have displacement coupling terms instead of velocity coupling terms and are in local forms:

$$\sigma = C : e + S \cdot u, \tag{2a}$$

$$\nabla \cdot \sigma + f = S^T : e + K \cdot u + \rho \cdot \ddot{u}, \tag{2b}$$

where $\boldsymbol{C}$ is the elasticity tensor; $\boldsymbol{S}$ is a third order tensor and $S^{T}: e$ denotes $S_{ijk}e_{ij}$ in index notation; $\boldsymbol{\rho}$ is the mass density tensor; and

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$\boldsymbol{K}$ is a symmetric matrix. However, Yao et al. (2018) proved that Eq. (2) is mathematically equivalent to Eq. (1) by using the differential property of time convolution. They further demonstrated that $\boldsymbol{S}$ is the gradient of pre-stresses and $\boldsymbol{K}$ is the negative gradient of the initial body force that balances with the pre-stresses. Since these local form equations contain the gradient of pre-stress, they can be regarded as the limits of non-local form equations when the group of inhomogeneous materials approaches to a point. These theoretical results are not only in accordance with the theoretical findings of Xiang and Yao (2016), but also have been verified by a rotational spring experiment (Yao et al., 2018).

Since the transformation method for the design of metamaterials involves only spatial mappings, the elastic wave should use the same time to pass through the transformed media in the physical space as that to pass through the homogeneous media in the virtual space. This means the accurate elastodynamic equations for inhomogeneous media should have the time synchronization property. Although this property has been numerically demonstrated by the example of two dimensional (2D) elastic wave rotators in Xiang (2014), it is better to give a more explicit theoretical proof by using analytical solutions. However, since it is very difficult to obtain analytical solutions for general 2D or three dimensional (3D) problems (see Appendix), people usually use the solution of a 1D problem to underline basic mechanisms (Willis, 2009). Following this logic, this paper will discuss only P wave propagations in 1D transformed inhomogeneous media and demonstrates that only the Willis-form equations with displacement coupling terms have the time synchronization property, while the classical elastodynamic equations are special cases of the Willis-form equations for homogeneous media. Thus, it gives another evidence of the validity of the Willis-form equations.

2. The 1D Willis-form equations

The 1D classical elastodynamic equations free of body force in the virtual space can be written as

$$\bar{\sigma}(\bar{x},t) = \bar{C}\frac{\partial \bar{u}(\bar{x},t)}{\partial \bar{x}}, \tag{3a}$$

$$\frac{\partial \bar{\sigma}(\bar{x},t)}{\partial \bar{x}} = \bar{\rho}\frac{\partial^2 \bar{u}(\bar{x},t)}{\partial t^2}, \tag{3b}$$

where $\bar{x}$ is the coordinate in the virtual space; t denotes time; $\bar{u}$ is the displacement; $\bar{\sigma}$ is the longitudinal stress; $\bar{C}$ and $\bar{\rho}$ are the constant stiffness and mass density, respectively. In this paper, field variables and constants in the virtual space are denoted with overhead bars; otherwise they are in the physical space. After applying a spatial mapping $x = f(\bar{x})$, Eq. (3) is expected to change the form.

Generally, Norris and Shuvalov (2011) have pointed out that one can obtain the Willis-form equations if assuming the field variables before and after transformation satisfy $u = F^{-\mathrm{T}} \cdot \bar{\mathbf{u}}$ and $\sigma = F \cdot \bar{\sigma} \cdot F^{\mathrm{T}}/\det(F)$, where F is the deformation gradient with elements $F_{ij} = \partial x_i/\partial \bar{x}_j$. Thus, for this special 1D problem one can assume

$$u(x,t) = \bar{u}(\bar{x},t)/f'(\bar{x}), \tag{4a}$$

$$\sigma(x,t) = f'(\bar{x})\bar{\sigma}(\bar{x},t), \tag{4b}$$

because $F = F^T = \det(F) = f'(\bar{x})$, where $f'(\bar{x})$ is the derivative of f over $\bar{x}$.

Substituting Eq. (4) into Eq. (3), and using the chain rule $\partial()/\partial\bar{x} = f'(\bar{x})\partial()/\partial x$, one can obtain the 1D Willis-form equations in the physical space

$$\sigma(x,t) = C(x)\frac{\partial u(x,t)}{\partial x} + S(x)u(x,t), \tag{5a}$$

$$\frac{\partial \sigma(x,t)}{\partial x} = S(x)\frac{\partial u(x,t)}{\partial x} + K(x)u(x,t) + \rho(x)\frac{\partial^2 u(x,t)}{\partial t^2}, \tag{5b}$$

where

$$C(x) = \bar{C}\left[f'(\bar{x})\right]^3, \quad \rho(x) = \bar{\rho}f'(\bar{x}),$$
$$S(x) = \bar{C}f'(\bar{x})f''(\bar{x}) \ \text{ and } \ K(x) = \bar{C}\left[f''(\bar{x})\right]^2/f'(\bar{x}). \tag{6}$$

Compared with Eq. (3), Eq. (5) has three additional terms: Su, $S\partial u/\partial x$ and Ku, which are generally nonzero in inhomogeneous media if $f''(\bar{x}) \neq 0$. If the spatial mapping function f is linear, the transformed media must be homogeneous. In this case, Eq. (5) is the same as the classical elastodynamic equation, because $f'(\bar{x}) = const$ and $f''(\bar{x}) = 0$.

Eq. (5a) is the Willis-form constitutive equation that defines the stress increment due to an infinitesimal deformation in inhomogeneous media. It is different from the classical constitutive equation in homogenous media due to an additional term Su, which contains the gradient of pre-stress S. This is a natural result because the pre-stress is the concomitant of inhomogeneity (Xiang and Yao, 2016; Yao et al., 2018). The similar pre-stress effects due to the transformation are also reported by Norris and Parnell (2012) and Colquitt et al. (2014).

As aforementioned, the classical elastodynamic equations are commonly used to approximately describe the wave propagation in inhomogeneous media. By removing the additional terms in Eq. (5), one can obtain the classical 1D elastodynamic equations for the inhomogeneous transformation media

$$\sigma(x,t) = C(x)\frac{\partial u(x,t)}{\partial x}, \tag{7a}$$

$$\frac{\partial \sigma(x,t)}{\partial x} = \rho(x)\frac{\partial^2 u(x,t)}{\partial t^2}. \tag{7b}$$

3. Wave velocity

Substituting Eq. (5a) into Eq. (5b), yields

$$C(x)u_{,xx}(x,t) + C_{,x}(x)u_{,x}(x,t) + R(x)u(x,t) = \rho(x)\frac{\partial^2 u(x,t)}{\partial t^2}, \tag{8}$$

where the subscript ‘,’ denotes the differential operation over x; and $R(x) = S_{,x}(x) - K(x)$.

According to the Bloch’s theorem (Brillouin, 1946), the wave amplitude $a(x)$ should be considered as a function of position for periodic media. Since periodic media can be regarded as special cases of inhomogeneous media, one can try to extend this assumption for general inhomogeneous media. Furthermore, the initial wave phase should be calculated as an integration of wave number $p(x)$ along the route from a reference position x_0 with zero phase to the present position x. Therefore, the displacement of monochromatic vibrations in inhomogeneous media can be written as

$$u(x,t) = a(x)\exp\left\{\mathrm{i}\left[\int_{x_0}^{x} p(s)\mathrm{d}s - \omega t\right]\right\}, \tag{9}$$

where ω is the angular frequency. This equation is also correct for homogenous media, where the wave amplitude and wave number are constants.

It is understood that the complex notation of Eq. (9) is a convenient choice for mathematical deduction. The physical quantity is either the real or the imaginary part of the corresponding solution. Therefore, after substituting Eq. (9) into Eq. (8), the resultant real and imaginary parts should both equal to zero:

$$\left[R(x) + \rho(x)\omega^2 - C(x)p^2(x)\right]a(x) + C_{,x}(x)a_{,x}(x) + C(x)a_{,xx}(x) = 0, \tag{10}$$