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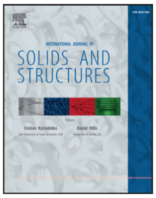

On the elastic far-field response of a two-dimensional coated circular inhomogeneity: Analysis and applications

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ABSTRACT

The paper studies the conditions under which the far-field response of a two-dimensional coated circular inhomogeneity embedded into an infinite matrix and subjected to uniform stresses at infinity is identical to that of a perfectly bonded inhomogeneity. The problem is considered in plane strain and antiplane settings. All constituents of the composite systems are assumed to be isotropic or transversely isotropic (with the axis Oz in longitudinal direction) and linearly elastic. For the plane strain problem and hydrostatic load or antiplane problem, it is shown that there always exists an equivalent inhomogeneity of the radius equal to the external radius of the coating that produces the elastic fields inside the matrix that are identical to those of the original coated inhomogeneity. For the plane strain and deviatoric load, the elastic fields in the matrix due to these two composite systems are always different, except for the equal shear moduli case. However, it is rigorously proved here that, for the deviatoric load and any combination of the material parameters, there exists the equivalent inhomogeneity of the radius equal to the external radius of the coating that induces the same dipole moments as those induced by the coated inhomogeneity. The existence of the equivalent inhomogeneities whose radius is different from the external radius of the coating is also investigated. The application of the proposed procedure to the homogenization problems leads to the new closed-form expression for the effective transverse shear modulus of transversely isotropic unidirectional composites. The findings presented here provide an insight on the influence of the interphases that could be useful in the analysis of some types of inverse problems.

1. Introduction

When an inhomogeneity is inserted in an infinite homogeneous matrix subjected to uniform fields at infinity (e.g. temperature gradient, stress, etc.), the far-field response of the composite system changes as compared to that of homogeneous material. The coefficients involved in the far-field asymptotic expansions contain information on the inhomogeneity's shape and properties. This fact is used in one of the oldest homogenization schemes - Maxwell's (1873) approximation formula (see Milton, 2002, Torquato, 2002). For the composite materials reinforced with the fibers/particles perfectly bonded to the material matrix, the scheme provides the estimates for the properties of an equivalent homogeneous material (effective properties) that are consistent with or, for some properties, identical to those provided by the major effective medium theories such as composite cylinder/sphere model of Hashin and Hashin and Rosen, the Mori–Tanaka method, the Generalized Self-consistent Model, and to the Hashin–Shtrikman variational bounds (See Milton, 2002; Torquato, 2002, McCartney and Kelly, 2008, McCartney, 2010, Mogilevskaya et al., 2012).

In this paper, we investigate the influence of the coating layer on the far-field response of the materials reinforced with coated fibers of circular cross sections in plane strain and antiplane settings. The purpose of this paper is twofold. First, we compare the elastic fields inside the matrix induced by the coated inhomogeneity with those induced by the equivalent perfectly bonded inhomogeneity. The goal of this comparison is to identify the conditions under which the presence of the coating layer can be recorded via the induced elastic fields away from the coated fiber. Second, we apply the findings of the analysis to homogenization problems by formulating a new replacement procedure for the coated inhomogeneity and studying its links to various replacement procedures reported in the literature. The procedure consists in replacing the original problem of a single coated inhomogeneity by that of a single isotropic or transversely isotropic inhomogeneity perfectly bonded to the material matrix. The elastic properties of the

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equivalent inhomogeneity are obtained by comparing the far-field responses of the two composite systems.

The literature dealing with the problems of coated fiber/particle is extensive (See e.g. Walpole, 1978, Benveniste et al., 1989, Benveniste, 2014, Hashin, 1990, Hashin, 2002, Cherkaoui et al., 1994, Chu and Rokhlin, 1995,El Mouden et al., 1998, Sarvestani, 2003, Guinovart-Díaz et al., 2005, Kari et al., 2008, Bonfoh et al., 2012, Nguyen and Pham, 2014, Tran et al., 2017), and the references therein. Most of these publications are concerned with the overall response of the composites with coated reinforcements and some of them employ the so-called replacement concept that consists in replacing the original problem of a single coated inhomogeneity by that of a single isotropic or transversely isotropic inhomogeneity perfectly bonded to the material matrix. The effective properties of the composite material are then obtained with the use of the schemes developed for the perfectly bonded inhomogeneities.

The first replacement procedure can be attributed to Hill (1964) who studied the equivalent properties of a "cylindrical composite element." Applying axisymmetric loads and using two different approaches (based on either analysis of the elastic fields in the element or energy considerations), Hill showed that a circular coated fiber can be replaced by an equivalent homogeneous fiber. Another replacement method proposed by Hashin (1990), Hashin (2002) is based on the use of composite sphere/cylinder model of Hashin (1962) and Hashin and Rosen (1964), (See e.g. Qiu and Weng, 1991, Chu and Rokhlin, 1995). While this procedure leads to the closed-form expressions for some moduli (e.g. bulk modulus), it provides only bounds on the remaining moduli. In addition, the expressions for these bounds are very involved. To bypass this problem Tran et al. (2015, 2017) used the replacement procedure combined with either dilute solution scheme or using Maxwell-type approximation for the equivalent inhomogeneity moduli. However, some parameters involved in the final expressions for the effective transverse shear modulus "*are unfortunately too big to be given explicitly*." While the authors provided a simplified empirical closed-form expression for the effective transverse shear modulus (that has the same structure as the expression for the effective transverse two-dimensional bulk modulus), its validity for any combination of material parameters has not been investigated. Nguyen and Pham (2014) provided the closed-form expressions for the case of incompressible media (also without the analysis of its validity for all combinations of material parameters). Another closed-form expression given in Duan et al. (2007) contains the parameter that has not been given explicitly and, according to the authors' statement, is lengthy. The validity of the latter expression for any combination of material parameters has not been investigated as well.

The interest in various replacement procedures was revived by recent activities in the area of modeling of nano materials (See Duan et al., 2005, Duan et al., 2007, Chen et al., 2007, Kari et al., 2008, Gu et al., 2014, Benveniste, 2014, Nazarenko and Stolarski, 2016, Xu et al., 2016), and the references therein. In most of these papers, the interphase is assumed to be thin. Therefore, it can either be replaced by an imperfect interface models derived by asymptotic analysis (See e.g. Benveniste and Miloh, 2001, Gu et al., 2014, Xu et al., 2016), or analyzed under the assumption of the uniform strains within the inhomogeneity. The latter assumption works well for axisymmetric loading, which explains why most of the methods advocated in these publications lead to identical expressions for the effective bulk modulus. However, the assumption is certainly not valid for the general loading case.

In this paper, we analyze the replacement concept that is based on the comparison of the far-field response of a two-dimensional coated circular inhomogeneity embedded into an infinite matrix and subjected to uniform stresses at infinity with that of a perfectly bonded inhomogeneity. No restrictions are made on either thickness of the interphase (whose properties are different to those for the matrix) or on the structure of the strain fields within the system. The elastic fields away from the coated inhomogeneity are obtained exactly by solving the corresponding plane strain or antiplane elasticity problems.

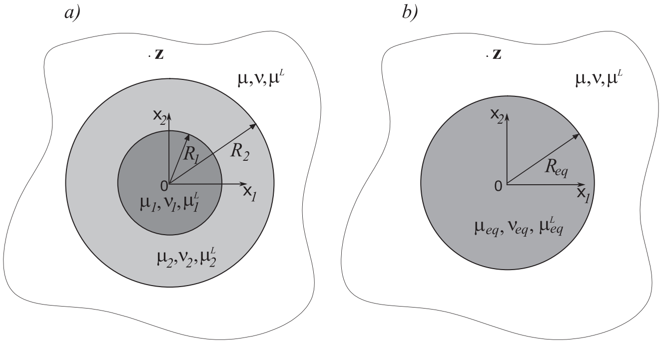

Fig. 1. (a) Problem I - a coated inhomogeneity in an infinite plane and (b) Problem II - an equivalent inhomogeneity in an infinite plane.

For the case of hydrostatic or antiplane load, it is shown that there always exists an equivalent inhomogeneity of the radius equal to the external radius of the coating that produces the same elastic fields inside the matrix as the original coated inhomogeneity. This conclusion is in agreement with previous observations of several authors who employed various energy- or uniform fields-based considerations. For the deviatoric load, it is proved that the elastic fields in the matrix phase for the two composite systems are always different, except for the equal shear moduli case. However, we prove that, for *any combination of the material parameters*, there exists the equivalent inhomogeneity of the radius equal to the external radius of the coating that induces the same dipole moments as those induced by the coated inhomogeneity. This proof constitutes the major contribution of the present study. In addition, we investigate the conditions for the existence of the equivalent inhomogeneity whose radius is different from that of the coating and provide some examples of the conditions under which the coated inhomogeneity does not disturb the fields inside the matrix under specific types of loadings at infinity. The proposed analysis has applications in homogenization problems for transversely isotropic unidirectional composites where it leads to the closed-form expressions for the relevant effective properties, especially for the transverse effective shear modulus; as stated above, the previously published estimates are either lengthy or not valid for some combinations of material properties. Moreover, the findings presented here are of more general nature. They provide an insight on the influence of the interphases on the far-field response of the materials and could be useful in the analysis of some types of inverse problems.

2. Problem formulation

Consider two problems illustrated in Fig. 1. The composite system of the first problem, Fig. 1a, consists of an infinite isotropic elastic matrix containing an embedded coated circular inhomogeneity. Let R_1, R_2 be the radii of the concentric circles that serve as the internal and external boundary of the coating, respectively. Assume that the centers of the circles are located at the origin of the Cartesian coordinate system. The second composite system, Fig. 1b, consists of an infinite isotropic elastic matrix containing a single perfectly bonded elastic circular inhomogeneity of radius R_{eq}, with the center also at the origin of the coordinate system. The corresponding relevant properties of the inhomogeneity and the coating of Fig. 1a are μ_1, ν_1 and μ_2, ν_2, respectively (plane strain problem)

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