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Loading paths for an elastic rod in contact with a flat inclined surface

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ABSTRACT

This paper computes stationary profiles of an isotropic, homogeneous, linearly elastic rod with its end-point locations and tangents specified. One end of the rod is clamped and the other end makes contact with a flat, rigid, impenetrable surface, which is displaced towards the clamped end. This boundary value problem has applications to biomechanical sensory devices such as mammal whiskers. The paper gives exact analytical solutions to the boundary value problem, embracing the planar equilibrium configurations for both point contact and line contact with the wall. Plots of loading paths for different inclinations of the wall provide an insight into the force-displacement relationship appertaining to real world slender rods under this type of loading. This report is complemented by data obtained from corresponding experimental studies which shed light on the differences between the model, which is based on the mathematical theory of elasticity, and the mechanics of real world long slender bodies, such as mammalian vibrissal systems.

1. Introduction

The mechanics of a long elastic rod or bar that is clamped at one end and has a load applied at the other end arises in many areas of structural engineering; for example, it finds wide application in relation to the critical load that a strut or column can sustain prior to buckling, i.e., ‘Euler Buckling’. It additionally arises within the context of biomechanics. For example, animal whiskers and antennae can be characterised as long, slender, flexible rods, which are fixed at one end whilst the other end is free to undergo large deflections under applied axial loadings, see Birdwell et al. (2007), and Lungarella et al. (2002). In the case of certain mammals, for example rats, that load-deflection relationship provides information about their surroundings, i.e., the whisker is a sensory device, connected to its neurological system. That form of tactile sensing has attracted considerable interest from researchers in robotics and neuroscience, see for example Solomon and Hartmann (2006) and Mitchinson and Prescott (2013). If pressed further against a surface, a section of the whisker tends to establish line contact with the surface, providing further information on the shapes and textures of objects, see Dehnhardt (1994). Similar line contact problems arise in the mechanics of peeling flexible adherends and have been studied by Majidi et al. (2005). The problem of an extensible rod peeled off a flat sticky surface is the focus of a study by He et al. (2013), whilst Wu et al. (2015) consider the adhesion of the Tokay gecko, a creature whose remarkable climbing abilities are attributable to its sticky feet.

The load–deflection relationship in those point contact and line contact problems depends on properties of the rod, including its flexural rigidity, its mass per unit length, its extensibility and its intrinsic curvature. Furthermore, all of those aforementioned properties may vary with length and may additionally depend on the orientation of the rod due to anisotropy. Studies on rats, such as those by Towal et al. (2011) and Voges et al. (2012), show that taper is important and has certain biomechanical advantages. Kulikov (2013) has additionally found similar advantages in Russian Desman whiskers, and Ginter Summarell et al. (2015) in pinniped whiskers. In one study, by Towal et al. (2011), the geometric configuration of a whisker is modelled as a parabola. Birdwell et al. (2007) include initial curvature and taper, but their model involves a linearisation of the exact expression for curvature, which strictly applies only to small deflections of a whisker. However, in formulating models that aim to provide insight into the mechanics of whiskers, it is convenient to establish a benchmark model against which the effects of initial curvature, taper, weight and so forth can be interpreted. In the mathematical theory of elasticity, the elastica model assumes that the internal bending moment of an unshearable, inextensible rod is linearly proportional to its curvature, see Antman (2005). Since the elastica involves the exact (nonlinear) expression for curvature it applies to both large and small deflections. It establishes the basis for studies of whiskers by Clements and Rahn (2006). However, the authors

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do not represent the problem as a boundary value problem, nor do they solve the equations analytically.

This paper applies the elastica model, formulated as a boundary value problem, to the mechanics of both point and line contact. The rod/whisker is assumed to be isotropic and homogeneous, i.e., constant elastic stiffness along its length. Given that stiffness is more important than weight, the latter is ignored. Additionally, the rod is assumed to be inextensible and straight in its natural unstressed configuration.

Regarding the boundary conditions, we examine the equilibrium of a rod/whisker that has one end fixed (in the mammal's face) and the free end deflects upon contact with an inclined wall that is displaced towards the fixed end such that it compresses the rod/whisker. The problem of determining the configuration of a rod in point contact with an inclined load has been studied before, for example by Frisch-Fay (1962) and Navaee and Elling (1992). The analysis presented here unifies the mechanics of point contact with line contact. It additionally covers a range of angles of inclination of the wall, from the vertical to the horizontal. It presents exact analytical solutions and accompanying plots that illustrate the relationship between the displacement of the wall and the corresponding compressive force exerted on the rod. The analysis is complemented by data from experiments on slender, flexible nickel–titanium rods, which shed light upon issues arising with respect to the behaviour of similar rod-like structures, such as whiskers.

The paper is set out as follows: The next section specifies the experimental set-up, the physical boundary conditions and the parameterisation of the rods used in the experiments. In the section after that, the mathematical model is expressed as a dimensionless system of six first-order nonlinear ordinary differential equations. That is followed by a full solution to the boundary value problem, involving Jacobian elliptic integrals (refer to Abramowitz and Stegun (1966) for information on those integrals and the associated elliptic functions). The penultimate section presents force–displacement loading paths of experimental data together with those predicted by elastica theory. The paper ends with a discussion of its findings within the context of further studies on whiskers and real-world problems generally.

2. Formulation of the boundary value problem

Formulated during the eighteenth century by Euler and Bernoulli, the elastica is an established model for the large deflections of long slender rods; see Levien (2008) and Goss (2009) for historical perspectives. Consequently, the formulation of the boundary value problem set out in this paper follows a well trodden path, but we mention here Frisch-Fay (1962) and Batista (2013) where we find related formulations, and the constrained problems considered in Plaut et al. (1999) and Domokos et al. (1997).

Nickel–titanium alloy rods of circular cross-section with radius 0.5 mm and lengths varying from 300 to 500 mm were selected for the experiments. We assume, with good justification, that the rods are inextensible, unshearable, isotropic and homogeneous. Each rod has length L and is parameterised by the independent arc-length variable S, where $0 \leq S \leq L$. In its unstressed condition the rod lies straight, i.e., it has no intrinsic curvature. That straight state is the reference state from which all experiments begin. It also marks out the X axis, i.e., in its natural state the rod lies along the X axis.

The rod's bent form is planar. Its configuration is specified by the coordinates $X(S)$, $Y(S)$ and an angle $\psi(S)$, measured anticlockwise from the X axis, see Fig. 1, with

$$\frac{dX}{dS} = \cos\psi, \tag{1}$$

$$\frac{dY}{dS} = \sin\psi. \tag{2}$$

The curvature Γ of the rod is expressed in terms of the change in slope:

$$\frac{d\psi}{dS} = \Gamma. \tag{3}$$

In an experiment one end of the rod, designated $S = 0$, is fixed at zero in a chuck such that its position $X(0)$ and $Y(0)$ and its slope $\psi(0)$ are fixed throughout the experiment. Those boundary conditions can be expressed as

$$X(0) = 0, \tag{4}$$

$$Y(0) = 0, \tag{5}$$

$$\psi(0) = 0. \tag{6}$$

An experiment proceeds by displacing a rigid plane surface, referred to as a 'wall', along the X axis from $S = L$ towards $S = 0$, by amount D, as shown in Fig. 1. The wall is oriented to the X axis at an angle $\pi/2 - \alpha$, where $0 \leq \alpha \leq \frac{\pi}{2}$. We assume that friction between the wall and the rod is negligible. As D increases, the tip of the rod ($S = L$) is in point contact with the wall and deflects upwards along the wall until a point is reached whereby the tangent at the tip is parallel with the wall, i.e., $\psi(L) = \pi/2 - \alpha$. That occurs at a value of D denoted D_c. Upon further increases in displacement of the wall, $D > D_c$, the free end of the rod slides further up the wall, such that a section of rod, of length B, is in line contact with the wall. The remaining 'free' section of rod is of length $L - B$. The computation of D is with respect to that free length, as follows:

$$D = L - X(L) + Y(L)\tan\alpha, \quad \text{for point contact}, \tag{7}$$

$$D = L - X(L-B) + Y(L-B)\tan\alpha, \quad \text{for line contact}. \tag{8}$$

That set-up involves the following boundary conditions:

$$X(L) = L - \Delta, \quad \text{for} \quad D < D_c. \tag{9}$$

$$X(L-B) = L - \Delta, \quad \text{for} \quad D > D_c. \tag{10}$$

where Δ is the 'end shortening' measured to the point of first contact with the wall, i.e., for $D < D_c$ it is the amount of horizontal displacement of the tip of the rod $S = L$, and for $D > D_c$, Δ it is the horizontal displacement of the point $S = L - B$.

During an experiment, the end of the free section of the rod is in contact with the wall. The curvature (Γ) of that free section of bent rod does not change sign along its length. However, since no external bending moments are applied at the point of contact with the wall, the curvature at that point is zero and the the following holds:

$$\Gamma(L) = 0, \quad \text{for} \quad D < D_c, \tag{11}$$

$$\Gamma(L-B) = 0 \quad \text{for} \quad D > D_c, \tag{12}$$

Assuming external moments and forces, e.g., weight, are negligible, the loads acting on each element of the rod can be decomposed into a compressive force with magnitude denoted T, acting parallel with the X axis and in the negative direction, plus a force acting along the Y axis in the positive direction, with magnitude denoted N. There is also a bending moment, which acts anticlockwise about an axis normal to the X, Y plane, of magnitude M. The forces N and $-T$ can be expressed in terms of R, the force acting normal to the wall, where $R^2 = N^2 + (-T)^2$, see Fig. 2. From