International Journal of Solids and Structures 000 (2016) 1–23

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Post-bifurcation and stability of a finitely strained hexagonal honeycomb subjected to equi-biaxial in-plane loading

Christelle Combescure[a,b], Pierre Henry[a,b], Ryan S. Elliott[a,*]

[a] Department of Aerospace Engineering and Mechanics, University of Minnesota, Minneapolis, MN 55455, USA
[b] Laboratoire de Mecanique des Solides (CNRS UMR 7649), Ecole Polytechnique, Route de Saclay, 91128 Palaiseau, France

ARTICLE INFO

Article history:
Received 27 October 2015
Revised 9 December 2015
Available online xxx

Keywords:
Bifurcation
Honeycomb structures
Post-buckling
Stability

ABSTRACT

The buckling and crushing mechanics of cellular honeycomb materials is an important engineering problem. Motivated by the pioneering experimental and numerical studies of Papka and Kyriakides (1994, 1999a,b), we review the literature on finitely strained honeycombs subjected to in-plane loading and identify two open questions: (i) How does the mechanical response of the honeycomb depend on the applied loading device? and (ii) What can the *Bloch wave representation* of all bounded perturbations contribute to our understanding of the stability of post-bifurcated equilibrium configurations? To address these issues we model the honeycomb as a two-dimensional infinite perfect periodic medium. We use analytical group theory methods (as opposed to the more common, but less robust, imperfection method) to study the honeycomb's bifurcation behavior under three different far-field loadings that produce (initially) the same equi-biaxial contractive dilatation. Using an FEM discretization of the honeycomb walls (struts), we solve the equilibrium equations to find the principal and bifurcated equilibrium paths for each of the three loading cases. We evaluate the structure's stability using two criteria: *rank-one convexity* of the homogenized continuum (long wavelength perturbations) and Bloch wave stability (bounded perturbations of arbitrary wavelength). We find that the post-bifurcation behavior is extremely sensitive to the applied loading device, in spite of a common principal solution. We confirm that the *flower mode* is always unstable, as previously reported. However, our (first ever) Bloch wave stability analysis of the post-bifurcated equilibrium paths shows that the flower mode is stable for all sufficiently short wavelength perturbations. This new result provides a realistic explanation for why this mode has been observed in the finite size specimen experiments of Papka and Kyriakides (1999a).

1. Introduction

The buckling and crushing mechanics of cellular honeycomb materials subjected to in-plane loading has been a long-standing problem of practical importance to the engineering community, primarily due to their wide use in shock absorption and mitigation. The theoretical interest in this problem stems from the fact that it represents a prototypical example of nonlinear mechanical systems exhibiting instabilities that lead to strain localization and quasi-static propagating *shear-bands*. The excellent monograph by Gibson and Ashby (1997), now in its second edition, provides a thorough presentation, from the materials science viewpoint, of what is known about the behavior of such materials.

From the wide literature on the subject, the review of which is beyond the scope of this work, attention is focused here on the two-dimensional problem of in-plane crushing of honeycombs. Detailed studies have appeared on this topic in the mechanics literature, both experimental (Papka and Kyriakides, 1994, 1998, 1999a; 1999b, among others) and theoretical (Ohno et al., 2002; Okumura et al., 2002; Saiki et al., 2005, 2002; Triantafyllidis and Schraad, 1998). However, many open questions remain about the particular mechanisms by which honeycomb materials progressively buckle in a sequence that ultimately makes up their highly complex mechanical response to in-plane loading. Moreover, there seems to be some ambiguity in the literature regarding the application and effect of different mechanical loading devices as well as ambiguity in regard to the stability of the bifurcated equilibrium paths.

In this paper, we first review the relevant literature with a focus on the onset of instability, initial post-bifurcation and the stability of the resulting bifurcated equilibrium paths in finitely strained cellular solids with hexagonal symmetry under different in-plane loading conditions. Next, we investigate the behavior of an infinite, perfect, two-dimensional hexagonal honeycomb subjected to equi-biaxial compression from multiple loading devices. We use

* Corresponding author. Tel.: +1 6126242376.
E-mail address: relliott@umn.edu (R.S. Elliott).

http://dx.doi.org/10.1016/j.ijsolstr.2016.02.016
0020-7683/© 2016 Elsevier Ltd. All rights reserved.

analytical symmetry group theory methods (as opposed to the more common, but less robust, imperfection method) to study the honeycomb's bifurcation behavior under three different far-field loading conditions that produce (initially) the same equi-biaxial contractive dilatation: displacement control, dead load Biot stress control, and live load pressure control. Using the primitive unit cell (smallest possible periodic domain) and an FEM discretization of the honeycomb walls (struts), we solve the associated equilibrium equations, identify the principal path and bifurcation points along this path. Each bifurcation point corresponds to a loss of symmetry (either rotational, translational, or both) along the bifurcating equilibrium paths and group theory methods identify an appropriate primitive unit cell for each such path. Using the appropriate unit cell and the corresponding FEM mesh, we solve the associated equilibrium equations and compute the post-bifurcation behavior for the honeycomb in each of the three loading cases. For all obtained equilibrium configurations, we evaluate the structure's stability using two criteria: *rank-one convexity* of the homogenized continuum (for long wavelength perturbations) and Bloch wave stability (for bounded perturbations of arbitrary wavelength).

The presentation is organized as follows: in Section 2 we review the recent literature on the theoretical and numerical modeling of instability and bifurcation of honeycombs subjected to in-plane loading. In Section 3 we describe our theoretical and numerical model of a perfect, infinite, two-dimensional hexagonal honeycomb structure and the various loading devices considered. In Section 4 we present the equilibrium and stability problem of interest and discuss the symmetry-based branch-following and bifurcation techniques employed in the numerical (FEM based) solution of the nonlinear equilibrium equations. In Section 5 we report the results obtained from our simulations, and finally in Section 6 we conclude with a critical discussion and perspectives on the obtained results.

2. Previous investigations of the in-plane loading of honeycombs

The study of cellular solids has been a major topic of research in mechanics over the past few decades. However the analysis of the stability and post-bifurcation behavior of regular honeycomb structures has only been addressed relatively recently. The following is a short bibliographic review of papers which are relevant to the current study.

2.1. Experimental characterization

The investigation by Papka and Kyriakides (1994) on the uniaxial in-plane crushing of hexagonal aluminum honeycomb was the first in-depth study on this topic from the mechanics standpoint. These authors established the link between a bifurcation on the principal solution and the onset of localization due to the subcritical nature of the post-bifurcated equilibrium path (i.e. a reduction in load carrying capacity of the structure with increasing bifurcation amplitude). They also showed how the repetition of the above mechanism explains the sequential crushing of the cell rows, leading to the plateau behavior in the load–deformation response, which is the most characteristic (and technologically useful) aspect of the nonlinear behavior of these materials.

The subsequent work of Papka and Kyriakides (1999a) is an experimental study of the biaxial crushing of circular cell honeycombs and provides one of the best available accounts of the biaxial, in-plane loading behavior of hexagonal honeycomb materials. The authors use a custom made testing facility to crush polycarbonate honeycomb specimens of size 18 × 21 unit cells, investigating several bi-axiality ratios. The experiments aim to apply affine displacement-control boundary conditions to the specimen[1]:

$$\mathbf{x} = \mathbf{F}\mathbf{X}, \forall \mathbf{X} \in \partial\Omega, \tag{1}$$

where Ω is the body of the specimen, $\partial\Omega$ its boundary, and

$$\mathbf{F} = \begin{bmatrix} F_{xx} & 0 \\ 0 & F_{yy} \end{bmatrix}, \tag{2}$$

is the form of the deformation gradient (with respect to a Cartesian coordinate system aligned with the frame of the testing facility) used to describe the affine boundary conditions that the facility is capable of prescribing. Figs. 1 and 2 show the deformation sequence of the honeycomb material under equi-biaxial compression ($F_{xx} = F_{yy} < 1$). The initial response of the honeycomb is elastic up to the onset of instability where a *flower-like* mode develops. Much of the specimen behaves like the cluster of cells shown in Fig. 2, with a rather undeformed central cell and six surrounding highly deformed cells. This regular pattern is broken up in several places, due to a combination of three factors: edge effects, friction at the sliding boundary conditions and incompatibility of the mode of deformation with a finite specimen.

Figs. 3 and 4 show the deformation sequence of the honeycomb material with $F_{yy}/F_{xx} = 2$. The behavior of the honeycombs under these loading conditions is similar to the equi-biaxial compression. At the onset of instability, an almost flower-like mode develops. This mode is similar to the equi-biaxial flower-like mode but now the hexagonal C_6 symmetry of the mode is broken causing the entire flower-like pattern to experience a uniform deviation aligned with the principal axes of the applied affine boundary conditions. The pattern is not as uniform throughout the specimen as in the case of equi-biaxial compression, and the final collapse of the specimen spreads from these localized high-deformation regions.

Fig. 5 shows the localized deformation sequence of the honeycomb material with $F_{yy}/F_{xx} = 3$. Here the behavior is similar to that when $F_{yy}/F_{xx} = 2$, except that now a competing *rectangular* mode is observed in some regions of the specimen.

The experiments of Papka and Kyriakides (1999a) show that the in-plane biaxial crushing of circular cell honeycomb materials is a complex mechanical process involving an initial loss of stability associated with a flower-like or rectangular deformation mode followed by strain localization and propagation and finishing with densification of the material. These experimental results (as well as their previous work, Papka and Kyriakides, 1994; 1998) have inspired a number of theoretical studies that aim to explain the observed behavior associated with the onset of instability in such materials. These theoretical studies are reviewed next.

2.2. Characterization of the onset of failure

The paper by Triantafyllidis and Schraad (1998) performs a semi-analytical study of the onset of failure for infinite two-dimensional hexagonal cell honeycomb materials subjected to a

[1] In this paper we use the direct tensor notation of Tadmor et al. (2012). In this notation $\mathbf{x} = \mathbf{F}\mathbf{X}$ has the indicial form $x_i = F_{iJ}X_J$ (where summation on repeated indices is implied) and represents the contracted multiplication operation. A tensor product of a second-order tensor and a first-order tensor (vector) is denoted $\mathbf{T}\otimes\mathbf{a}$ with indicial form $T_{ij}a_k$. A centered dot between two vectors, as in $\mathbf{a} \cdot \mathbf{b}$, represents the inner product. Thus, the fully contracted product of a second-order tensor is written $\mathbf{a} \cdot (\mathbf{T}\mathbf{b})$ with indicial form $T_{ij}a_ib_j$. A double dot represents the inner product on second-order tensors, as in $\mathbf{T} : \mathbf{S}$ and has the indicial form $T_{ij}S_{ij}$. This notation is also extended to the fully contracted product of a fourth-order tensor $\mathbf{T} : \mathbf{K} : \mathbf{S}$ with indicial form $T_{ij}K_{ijkl}S_{kl}$.