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Conservation integrals for two circular holes kept at different temperatures in a thermoelastic solid

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ABSTRACT

An explicit analytic solution for thermal stresses in an infinite thermoelastic medium with two circular cylindrical holes of different sizes kept at different constant temperatures, under steady-state heat flux is presented. The solution is obtained by using the most general representation of a biharmonic function in bipolar coordinates. The stress field is decomposed into the sum of a particular stress field induced by the steady-state temperature distribution and an auxiliary isothermal stress field required to satisfy the boundary conditions on the holes. The variations of the stress concentration factor on the surface of the holes are determined for varying geometry of the holes. The concept of the conservation integrals J_k, M and L is extended to steady state thermoelasticity and the integrals are proved to be path-independent. These integrals are calculated on closed contours encircling one or both holes. The geometries of a hole in a half-space and an eccentric annular cylinder are considered as particular cases.

1. Introduction

In this paper, we focus on the problem of an infinite thermoelastic plane containing two circular holes of generally different diameters kept at different constant temperatures. Two aspects of the problem are discussed: (a) complete thermoelastic fields are obtained in closed explicit form and (b) path-independent integrals providing energy release rates due to pores translation, growth and rotation are derived in the form of contour integrals.

The study is motivated by a number of applied engineering problems ranging from the design of heat engines, nuclear plants and aircrafts to the enhancement of electronic devices and MEMS performance. Steady-state temperature distribution does not induce thermal stresses in simply connected bodies free to expand (Hetnarski and Eslami, 2009). Presence of any inhomogeneities (holes, inclusions, cracks), however, may lead to noticeable thermal stresses due to temperature gradient even for steady heat flow. It is one of the key factors appearing in the design of high-accuracy instruments and high-performance structures involving brittle materials, such as concrete or ceramics.

Despite their very attractive properties like high temperature continuous service ability, low thermal conduction and excellent resistance to wear and aggressive environment, ceramic materials do not exhibit any plastic deformation before failure and display a very small toughness to arrest cracks. A representative example of thermal failure of ceramic materials can be envisaged in ceramic thermal-barrier coatings used to protect turbine blades in aircraft and power plant engines. In such components, cracks usually nucleated at the cooling hole locations due to a combination of mechanical and thermal stresses (Mazur, et al., 2005; Kim et al., 2011).

Stress concentrations due to thermal loadings is also a critical factor for the accurate design of MEMS and microelectronic packages, where several electric connections are embedded in a silicon or ceramic matrix at very small distance from each other. In this case, the heat production due to the Joule effect may induce thermal stresses sufficient to cause cracking and rupture of the insulating ligaments between the connections (Hsueh and Evans, 1985; Suhir, 2009).

Solution for the two-dimensional problem of the stress distribution in an infinite elastic medium containing two circular holes is based on the results of Jeffrey(1921) who found the most general form of a biharmonic stress function in bipolar coordinates and separated out the terms which give rise to multiple-valued displacement. Jeffrey (1921) illustrated his method solving some problems using Fourier series expansion of the Airy stress function in bipolar coordinated. However, he did not consider the problem of an infinite plate with two unequal circular holes.

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The problem of thermal loading of an infinite plate containing two holes was first studied by Van der Linden (1956) who considered two circular holes of the same radius subjected to different temperatures at their contours and Florence and Goodier, (1959) who considered a circular hole in a half-plane. Then, Chattarji and Ghosh (1969), Iwaki and Miyao (1973) and Iwaki (1973) calculated the distribution of thermal stress in an infinite plates with two circular holes under various thermal loading conditions. In particular, Iwaki and Miyao (1973) investigated the thermal stresses in a plate containing two insulated circular holes of different sizes under uniform heat flow in an arbitrary direction, whereas Iwaki (1973) solved the same thermoelastic problem here considered by using bipolar coordinates but a slightly different approach. Bipolar coordinates are used more recently, Wu and Markenscoff (1997) examined the asymptotic behavior of the stress in the ligament between two holes of different radii in an infinite plate under thermal loading. This paper focuses mostly on the stress singularity within the ligament between the holes, so that the solution retains insufficient number of terms to satisfy the most general boundary conditions (such as vanishing of the remote stress field).

Radi and Strozzi (2009) used the solution to solve the problem of a circular disk containing a sliding eccentric circular inclusion. Later, Radi (2011) studied the problem of stresses concentration induced in an infinite plate with two unequal circular holes by remote biaxial loading and arbitrary pressures in the holes. This author obtained closed form solution for the stress and displacement fields in the plate and investigated the variations of the stress concentration factor (SCF) at the edge of the two holes for various geometries and loading conditions.

Davanas (1992) showed that the elastic interaction between two pressurized holes is always repulsive, regardless the sign of the surface tractions on the holes. Tsukrov and Kachanov (1997) noticed that the interaction effects may be maximal in a slightly perturbed non-symmetric configurations. Panasyuk and Savruk (2009) presented a brief review of the elastic interaction between two holes in a stretched plate.

Radi (2011) also calculated integrals J_k, M and L on a closed contour encircling both holes under general remote loading conditions and traction free holes' surfaces. These path independent integrals provide the energy release rates associated with translation, self-similar expansion and rotation of the holes, respectively (Kienzler and Herrmann, 2000). Their values vanish when evaluated along every closed path, provided that there is no singularity enclosed within the path. As expected, these integrals do not necessarily vanish if the integration path encloses some singularity or cavity.

In this paper, we derive an explicit analytical solution for the steady thermal stresses induced in a two dimensional thermoelastic solid with two unequal circular holes by a difference in the temperatures kept at the holes contours. Solution is obtained by using the bipolar coordinates and is based on the approach of Jeffrey (1921). The analysis is based on the decomposition of the stress field on (i) fundamental stress field due to the steady temperature distribution, which gives non vanishing tractions on the circular boundaries and hydrostatic stress field at infinity, and (ii) an auxiliary isothermal stress field required to satisfy the vanishing of remote stress field and tractions on the surface of the holes. The solutions for the problems on a hole in a half-plane and an eccentric annulus are also obtained as special cases. The results are used to calculate the integrals J_k, M and L on a closed contour encircling one or both holes, for traction free holes surfaces and vanishing remote loading conditions. The purpose of this calculation is to provide some basic understanding for the role played by the thermoelastic conservation laws in multiple defects interaction, under steady-state thermal loading.

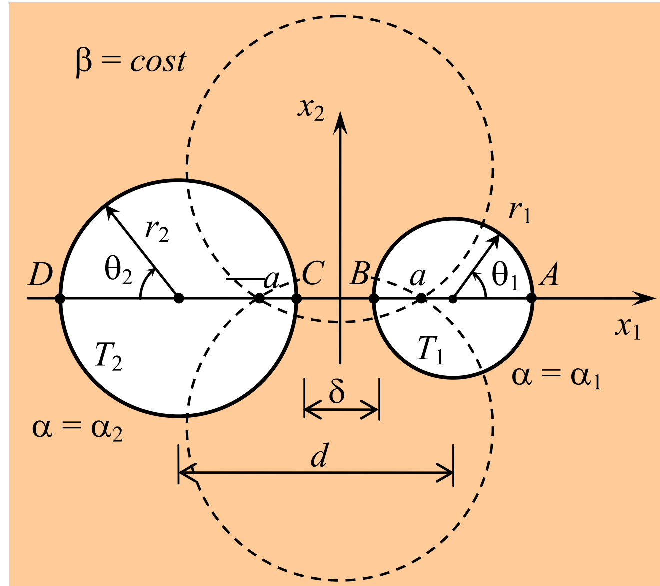

Fig. 1. Bipolar coordinate system for the problem of two unequal holes of radii r_1 and r_2, held at temperatures T_1 and T_2, respectively.

2. Problem description in bipolar coordinates

We consider two unequal circular holes subjected to thermal loading in an infinite medium under plane strain condition, or in a large plate under plane stress conditions (Fig. 1). Let r_1 and r_2 denote the radii of the two holes and d the distance between their centers. Temperatures T_1 and T_2 are prescribed on the edges of the holes and remote stress field is assumed vanishing. The occurrence of non vanishing tractions on the holes and remote stress field can be easily considered by adding the isothermal solution found by Radi (2011). Cartesian coordinate system $(0, x_1, x_2)$ is introduced as shown in Fig. 1. Following Jeffrey (1921), we use bipolar coordinates (α, β) defined by the following complex relation and its inverse complex mapping

$$z = a\,\coth(\bar{\zeta}/2) = a\frac{e^{\bar{\zeta}}+1}{e^{\bar{\zeta}}-1}, \quad \bar{\zeta} = \ln\frac{z+a}{z-a}, \tag{1}$$

where $z = x_1 + i x_2$ and $\xi = \alpha + i\beta$ are two complex variables and the upper bar denotes the complex conjugate. As shown in Fig. 1, the two poles of the bipolar coordinates are located on the x_1 axis at distance $\pm a$, with $a > 0$. Separation of real and imaginary parts in Eq. (1) leads to

$$x_1 = \frac{a\sinh\alpha}{\cosh\alpha - \cos\beta}, \quad x_2 = \frac{a\sin\beta}{\cosh\alpha - \cos\beta}. \tag{2}$$

The curves corresponding to constant values of α describe a family of coaxial circles with centers on the x_1 axis at distance a coth α from the origin and radius a/sinh α, whereas the curves corresponding to constant values of β describe another family of circles with centers on the x_2 axis and passing through both foci at $x_1 = \pm a$. All the geometrical parameters can be defined in terms of the radii of the holes and the distance between the centers of the holes. In particular, the surfaces of the holes are defined by $\alpha = \alpha_1 > 0$ and $\alpha = \alpha_2 < 0$, where

$$\alpha_1 = \operatorname{arccosh}\frac{r_1^2 - r_2^2 + d^2}{2\,d\,r_1}, \quad \alpha_2 = -\operatorname{arccosh}\frac{r_2^2 - r_1^2 + d^2}{2\,d\,r_2}. \tag{3}$$

With the reference to Fig. 1, the position of the poles along the x axis is given by

$$a = \frac{1}{2\,d}\sqrt{\left(d^2 - r_1^2 - r_2^2\right)^2 - 4\,r_1^2 r_2^2} = r_i \sinh|\alpha_i|, \quad (i = 1, 2) \tag{4}$$