International Journal of Solids and Structures 000 (2016) 1–9

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Mobility of a class of perforated polyhedra

Patrick W. Fowler[a,*], Simon D. Guest[b], Bernd Schulze[c,1]

[a] Department of Chemistry, University of Sheffield, Sheffield S3 7HF, UK
[b] Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, UK
[c] Department of Mathematics and Statistics, Lancaster University, Lancaster LA1 4YF, UK

ARTICLE INFO

Article history:
Received 4 November 2015
Revised 1 February 2016
Available online xxx

Keywords:
Rigidity
Mobility
Cubic polyhedron
Symmetry

ABSTRACT

A class of over-braced but typically flexible body-hinge frameworks is described. They are based on polyhedra with rigid faces where an independent subset of faces has been replaced by a set of holes. The contact polyhedron C describing the bodies (vertices of C) and their connecting joints (edges of C) is derived by subdivision of the edges of an underlying cubic polyhedron. Symmetry calculations detect flexibility not revealed by counting alone. A generic symmetry-extended version of the Grübler–Kutzbach mobility counting rule accounts for the net mobilities of infinite families of this type (based on subdivisions of prisms, wedges, barrels, and some general inflations of a parent polyhedron). The prisms with all faces even and all barrels are found to generate flexible perforated polyhedra under the subdivision construction.

The investigation was inspired by a question raised by Walter Whiteley about a perforated polyhedron with a unique mechanism reducing octahedral to tetrahedral symmetry. It turns out that the perforated polyhedron with highest ($\mathcal{O}_h$) point-group symmetry based on subdivision of the cube is mechanically equivalent to the Hoberman Switch-Pitch toy. Both objects exhibit an exactly similar mechanism that preserves $\mathcal{T}_d$ subgroup symmetry over a finite range; this mechanism survives in two variants suggested by Bob Connelly and Barbara Heys that have the same contact graph, but lower initial maximum symmetry.

1. Introduction

A trend in the treatment of mobility of frameworks composed of arrays of bodies connected by hinges is of the application of symmetry, wherever possible, to the counting of net mobility $m - s$, the balance of freedoms and constraints (or equivalently of mechanisms and states of self-stress) (Connelly et al., 2009; Fowler and Guest, 2000, 2002; Guest and Fowler, 2005, 2010; Guest et al., 2010; Röschel, 2012; Schulze et al., 2014; Schulze and Whiteley, 2011). One particular flexible framework realised as a Polydron™ model was described in a 2014 Fields Institute lecture by Walter Whiteley, at a meeting held to mark his 70th birthday; his observation of a symmetry-breaking mechanism of the model inspired the present investigation of an open-ended class of mobile frameworks based on the cubic polyhedra.

The basic object that sparked this investigation is W. Two further variants, R and B, emerged in discussions with Bob Connelly and Barbara Heys. In W six disjoint square faces of an octahedrally symmetric Archimedean polyhedron, the (small) rhombicuboctahedron (Cundy and Rollett, 1961), have been replaced by holes. B is also derived from this polyhedron. R is derived from the pseudo-rhombicuboctahedron discovered by Miller, as described in Rouse Ball and Coxeter (1987). All three objects are illustrated in Fig. 1. All exhibit a symmetry-breaking finite mechanism. Application of the established techniques for symmetry extension of mobility rules (Guest and Fowler, 2005) leads to an account of net mobility in all three structures. Interestingly, the explanation for the finite mechanism in W, which takes the structure from octahedral $\mathcal{O}_h$ to tetrahedral $\mathcal{T}_d$ symmetry, turns out to be identical with the symmetry account of the mechanism of the famous Hoberman Switch-Pitch toy (Chen et al., 2016; Hoberman, 2004)

The motivation for our symmetry treatment of an infinite class of structures is the initially surprising flexibility of some heavily over-constrained objects. W is an object with maximum octahedral rotational and reflectional symmetry belonging to the point group $\mathcal{O}_h$, which has 48 symmetry operations. Although over-braced by six states of self-stress according to simple counting, this framework has a mechanism that preserves the 24 symmetries of the tetrahedral $\mathcal{T}_d$ point group along a finite path that proceeds down from the high-symmetry point until a special geometry is

* Corresponding author. Tel.: +44 114 222 9538; fax: +44 114 222 9346.
E-mail addresses: p.w.fowler@sheffield.ac.uk (P.W. Fowler), sdg@eng.cam.ac.uk (S.D. Guest), b.schulze@lancaster.ac.uk (B. Schulze).
1 Supported by EPSRC First Grant EP/M013642/1.

http://dx.doi.org/10.1016/j.ijsolstr.2016.02.006

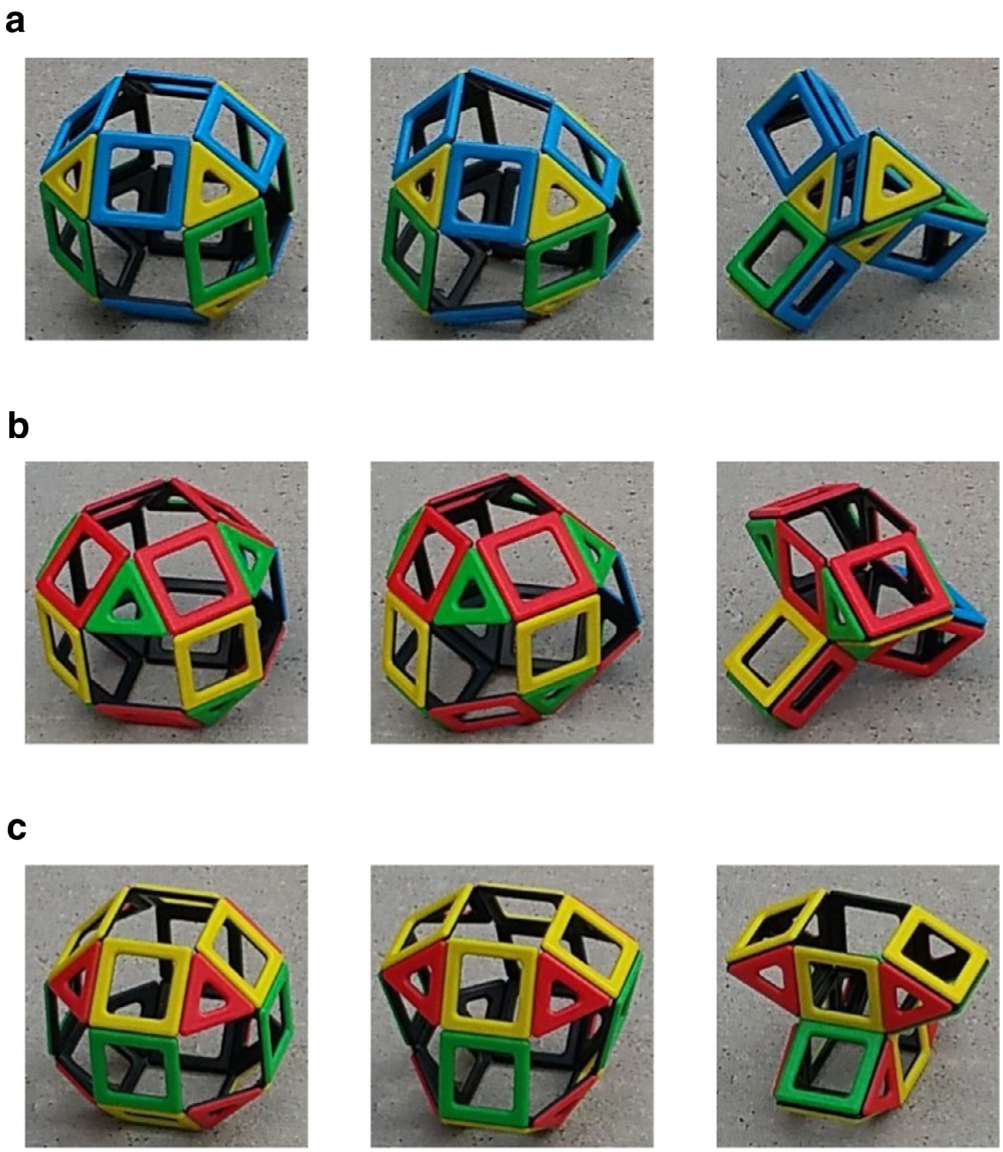

Fig. 1. Physical models of W, R and B, constructed from Magnetic Polydron™components. Rows (a), (b) and (c) correspond to W, R and B, respectively. Each row shows points on the path of the characteristic mechanism: initial high-symmetry configuration; the distortion mechanism, showing the halving of the symmetry group; the fully collapsed configuration after the pathway has passed through the multifurcation.

reached where multifurcation into lower symmetries takes place. The multi-branched pathway for further distortion starts at the point where each of four square faces becomes co-planar with its neighbours and can individually move radially in or out. Variants R ($\mathcal{C}_{4v}$) and B ($\mathcal{D}_{4h}$) show similar mechanisms that lead to halving of the symmetry group, with branching, and the possibility of further symmetry loss, at the co-planarity point or points. Fig. 1 shows snapshots along the path of the mechanism in W, R and B. In the following, we use the symmetry-extended mobility criterion to place the flexes of W, R and B in the context of infinite families of perforated polyhedra.

While we restrict attention to symmetric structures and their symmetry-induced mobility in this paper, we note that the mobility analysis of *generic* perforated polyhedral structures (without symmetry), under the term 'block-and-hole' polyhedra, is currently also an active area of research. In particular, it was shown in (Finbow-Singh and Whiteley, 2013) that under certain conditions, a generic embedding of a simplicial spherical polyhedron (which is rigid by Cauchy's rigidity theorem) remains rigid if a triangulated disc is cut out and new constraints are added into an essentially disjoint disc to create a rigid sub-structure (or rigid block). This result was very recently extended to structures with one rigid block and an arbitrary number of holes (Cruickshank et al., 2015). Moreover, it was shown in (Cruickshank et al., 2015; Finbow-Singh et al., 2012) that swapping the rigid blocks for holes and vice versa does not alter the rigidity properties of these perforated structures. The approach used here suggests that investigation of symmetry aspects of these general results for block-and-hole polyhedra and block-hole exchange would be a natural next step. This extension is currently in progress.

2. Symmetry-extended mobility criteria

The classic (Hunt, 1978) counting criterion for mobility (relative freedoms) $m - s$ of a mechanical linkage composed of n bodies connected by g joints is

$$m - s = 6(n-1) - 6g + \sum_{i=1}^{g} f_i, \qquad (1)$$