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# Numerical analysis of a viscoelastic mixture problem☆

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ABSTRACT

In this paper, we study, from the numerical point of view, a mixture problem involving a viscoelastic material and an elastic one. The mechanical problem is written as a linear system of two coupled hyperbolic partial differential equations. An existence and uniqueness result and an energy decay property are recalled. Then, fully discrete approximations are introduced by using the finite element method to approximate the spatial variable and the backward Euler scheme to discretize the time derivatives. A priori error estimates are proved from which, under suitable regularity conditions, the linear convergence of the algorithm is derived. Finally, some numerical simulations are presented to demonstrate the accuracy of the approximations and the behaviour of the solutions.



## 1. Introduction

During the last decades, the number of papers dealing with problems of mixtures including viscoelastic and elastic materials has increased considerably. Since the first studies of Truesdell and Toupin (1960), Kelly (1964) or Green and Naghdi (1965, 1968), many papers have considered mathematical issues as the existence and uniqueness of solutions or the qualitative properties of the solutions as, for instance, the energy decay property (see, e.g., Alves et al. 2009a, 2009b, 2013; Ieşan 1992, 2006; Ieşan and Quintanilla 2007; Rivera et al. 2013). One of the main applications of these models is the theory of the well-known composites, with an increasing use in the automotive industry.

In this paper, we revisit a problem involving a mixture of a viscoelastic material and an elastic one which was considered by Quintanilla (2005) within the theory of viscoelastic mixtures. He continued the work of Ieşan (2006), for a linear theory, where the dissipation effects were determined by the viscosity of rate type of a constituent and the relative velocity. Thermal effects were also included. Existence and uniqueness of a weak solution were proved by using results of the semigroup of linear operators theory, and an energy decay property was also shown. Moreover, limit cases, for instance corresponding to null viscosity coefficients, were analyzed.

Here, we assume the isothermal case and, for the sake of simplicity in the presentation, we restrict ourselves to the anti-plane shear deformations. We numerically study the problem, providing a priori error estimates, and we perform some numerical simulations to show the behaviour of the solutions.

The outline of this paper is as follows. In Section 2, we briefly describe the mathematical model and we introduce its variational formulation, for which an existence and uniqueness result, proved in Quintanilla (2005), is recalled. Then, fully discrete approximations are introduced in Section 3 by using a finite element method for the spatial approximation and the backward Euler scheme for the discretization of the time derivatives. An error estimate result is proved from which the linear convergence is deduced under suitable regularity assumptions. Finally, in Section 4 some numerical examples are shown to demonstrate the accuracy of the algorithm and the behaviour of the solution.

## 2. The model and its variational formulation

In this section, we present a brief description of the model (details can be found in Quintanilla (2005)).

Let $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, be a domain with a Lipschitz boundary $\Gamma = \partial\Omega$ decomposed into two disjoint parts $\Gamma_D$ and $\Gamma_S$ such that $\text{meas}(\Gamma_D) > 0$, and denote by $[0, T]$, $T > 0$, the time interval of interest.

Let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively. In order to simplify the writing, we do not indicate

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the dependence of the functions on $\boldsymbol{x}$ and $t$. Moreover, a dot above a variable represents its derivative with respect to the time variable.

Let $\tilde{u}, \tilde{w} \in \mathbb{R}^d$ be the displacement of the viscoelastic and elastic materials, respectively. Assuming the case of the isothermal processes, we consider a particular kind of solutions corresponding to anti-plane shear deformations. Then, we are interested in solutions of the form $\tilde{u} = (u_1, u_2, u_3) = (0, 0, u(x_1, x_2))$ and $\tilde{w} = (w_1, w_2, w_3) = (0, 0, w(x_1, x_2))$. We note that this assumption is done for the sake of simplicity in the presentation and the general case can be analyzed in a straightforward way.

Therefore, following Quintanilla (2005) the mechanical problem of a mixture involving a viscoelastic material and an elastic one is written as follows.

**Problem P.** Find the displacement of the viscoelastic material $u : \Omega \times [0, T] \to \mathbb{R}$ and the displacement of the elastic material $w : \Omega \times [0, T] \to \mathbb{R}$ such that

$$\rho_1 \ddot{u} = \alpha \Delta u + \beta \Delta w - \xi (u - w) + \mu^* \Delta \dot{u} - \xi^* (\dot{u} - \dot{w}) + \rho_1 F_1 \quad \text{in} \quad \Omega \times (0, T), \tag{1}$$

$$\rho_2 \ddot{w} = \beta \Delta u + \gamma \Delta w + \xi (u - w) + \xi^* (\dot{u} - \dot{w}) + \rho_2 F_2 \quad \text{in} \quad \Omega \times (0, T), \tag{2}$$

$$u = 0, \quad w = 0 \quad \text{on} \quad \Gamma_D \times (0, T), \tag{3}$$

$$u = 0, \quad w = 0 \quad \text{on} \quad \Gamma_S \times (0, T), \tag{4}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \quad w(\boldsymbol{x}, 0) = w_0(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in \Omega, \tag{5}$$

$$\dot{u}(\boldsymbol{x}, 0) = v_0(\boldsymbol{x}), \quad \dot{w}(\boldsymbol{x}, 0) = e_0(\boldsymbol{x}) \quad \text{for a.e. } \boldsymbol{x} \in \Omega. \tag{6}$$

Here, $\rho_1$ and $\rho_2$ are the mass densities of each constituent, $F_1$ and $F_2$ represent the body forces on each constituent, $\alpha$, $\beta$, $\xi$ and $\gamma$ are elastic coefficients, $\mu^*$ and $\xi^*$ denote viscoelastic coefficients, and $u_0, v_0, w_0, e_0$ are given initial conditions. Moreover, homogeneous Dirichlet boundary conditions have been also assumed on $\Gamma_S \times (0, T)$, but we point out that Neumann conditions could be used with some minor modifications in the analysis presented below.

In order to obtain the variational formulation of Problem P, let $Y = L^2(\Omega)$ and $H = [L^2(\Omega)]^d$, and denote by $(\cdot,\cdot)_Y$ and $(\cdot,\cdot)_H$ the respective scalar products in these spaces, with corresponding norms $\|\cdot\|_Y$ and $\|\cdot\|_H$.

Moreover, let us define the variational space $V$ as follows,

$$V = \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_D \cup \Gamma_S\},$$

with scalar product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$.

By using the classical Green's formula and boundary conditions (3) and (4), we write the variational formulation of Problem P in terms of the velocities $v = \dot{u}$ and $e = \dot{w}$. Here, $v$ denotes the velocity of the viscoelastic material and $e$ is the velocity of the elastic material.

**Problem VP.** Find the velocity of the viscoelastic material $v$: [0, $T$] $\to V$ and the velocity of the elastic material $e$: [0, $T$] $\to V$ such that $v(0) = v_0$, $e(0) = e_0$ and, for a.e. $t \in (0, T)$,

$$\begin{aligned}\rho_1 (\dot{v}(t), z)_Y &+ \xi (u(t) - w(t), z)_Y + \xi^* (\dot{u}(t) - \dot{w}(t), z)_Y \\ &+ \beta (\nabla w(t), \nabla z)_H + \alpha (\nabla u(t), \nabla z)_H + \mu^* (\nabla \dot{u}(t), \nabla z)_H \\ &= \rho_1 (F_1(t), z)_Y, \quad \forall\, z \in V,\end{aligned} \tag{7}$$

$$\begin{aligned}\rho_2 (\dot{e}(t), r)_Y &+ \xi (w(t) - u(t), r)_Y + \xi^* (\dot{w}(t) - \dot{u}(t), r)_Y \\ &+ \beta (\nabla u(t), \nabla r)_H + \gamma (\nabla w(t), \nabla r)_H \\ &= \rho_2 (F_2(t), r)_Y, \quad \forall r \in V,\end{aligned} \tag{8}$$

where the displacement of the viscoelastic material $u(t)$ is obtained from the relation

$$u(t) = \int_0^t v(s)\, ds + u_0, \tag{9}$$

and the displacement of the elastic material $w(t)$ is calculated from

$$w(t) = \int_0^t e(s)\, ds + w_0. \tag{10}$$

The existence and uniqueness of weak solutions as well as an exponential energy decay property have been considered in Quintanilla (2005), where the thermal effects were also included. It is stated in the following.

**Theorem 2.1.** *Assume that*

$$\mu^* > 0, \quad \xi > 0, \quad \xi^* > 0, \quad \alpha > 0, \quad \gamma > 0, \quad \rho_2 > 0, \quad \rho_1 > 0, \tag{11}$$

$$\gamma \alpha - \beta^2 > 0, \tag{12}$$

*and also*

$$F_1, F_2 \in C^1([0, T]; Y). \tag{13}$$

*Therefore, Problem VP has a unique solution with the regularity:*

$$u \in C^1([0, T]; H^1(\Omega)) \cap C^2([0, T]; Y),$$
$$w \in C([0, T]; H^1(\Omega)) \cap C^2([0, T]; Y).$$

*Moreover, if we define the energy function*

$$E(t) = \frac{1}{2} \int_\Omega (\rho_1 \dot{u}^2 + \rho_2 \dot{w}^2 + \xi (u - w)^2 + \alpha \nabla u \cdot \nabla u + \gamma \nabla w \cdot \nabla w + 2\beta \nabla w \cdot \nabla u)\, d\boldsymbol{x},$$

*and we assume that $F_1 = F_2 = 0$, then there exist two positive constants $M, \lambda > 0$ such that*

$$E(t) \le M E(0) e^{-\lambda t} \quad \text{for} \quad t \ge 0.$$

## 3. Fully discrete approximations: a priori error estimates

In this section, we now consider a fully discrete approximation of Problem *VP*. This is done in two steps. First, we assume that $\overline{\Omega}$ is a polyhedral domain and we consider a finite dimensional space $V^h \subset V$, approximating the variational space $V$, given by

$$V^h = \{v^h \in C(\overline{\Omega})\ ;\ v^h_{|_K} \in P_1(K) \quad K \in \mathcal{T}^h, \quad v^h = 0 \text{ on } \Gamma_D \cup \Gamma_S\}, \tag{14}$$

where $P_1(K)$ represents the space of polynomials of global degree less or equal to one in $K$ and we denote by $(\mathcal{T}^h)_{h>0}$ a regular family of triangulations of $\overline{\Omega}$ (in the sense of Ciarlet (1991)), compatible with the partition of the boundary $\Gamma = \partial\Omega$ into $\Gamma_D$ and $\Gamma_S$; i.e. the finite element space $V^h$ is composed of continuous and piecewise affine functions. Let $h_K$ be the diameter of an element $K \in \mathcal{T}^h$ and let $h = \max_{K \in \mathcal{T}^h} h_K$ denote the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $u^h_0$, $v^h_0$, $w^h_0$ and $e^h_0$, are given by

$$u^h_0 = \mathcal{P}^h u_0, \quad v^h_0 = \mathcal{P}^h v_0, \quad w^h_0 = \mathcal{P}^h w_0, \quad e^h_0 = \mathcal{P}^h e_0, \tag{15}$$

where $\mathcal{P}^h$ is the $L^2(\Omega)$-projection operator over $V^h$.

Secondly, we consider a partition of the time interval [0, $T$], denoted by $0 = t_0 < t_1 < \cdots < t_N = T$. In this case, we use a uniform partition of the time interval [0, $T$] with step size $k = T/N$ and nodes $t_n = n\,k$ for $n = 0, 1, \ldots, N$. For a continuous function $z(t)$, we use the notation $z_n = z(t_n)$ and, for the sequence $\{z_n\}_{n=0}^N$, we denote by $\delta z_n = (z_n - z_{n-1})/k$ its corresponding divided differences.

Therefore, using the backward Euler scheme, the fully discrete approximations are considered as follows.

**Problem VP$^{hk}$** Find the discrete velocity of the viscoelastic material $v^{hk} = \{v^{hk}_n\}_{n=0}^N \subset V^h$ and the discrete velocity of the elastic material $e^{hk} = \{e^{hk}_n\}_{n=0}^N \subset V^h$ such that $v^{hk}_0 = v^h_0$, $e^{hk}_0 = e^h_0$ and, for