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Exact degenerate scales in plane elasticity using complex variable methods

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ABSTRACT

A recent work has shown that using conformal mapping can lead to exact values of the degenerate scales in plane elasticity. We elaborate on this work by introducing some algebraic tools when this conformal mapping is a rational fraction transforming the outside of the unit circle into the outside of the considered domain. Using these tools, new cases are solved including shortened hypotrochoid, arc of circle, new approximates of equilateral triangle and square or symmetric Joukowski profiles. Another method makes it possible to obtain the degenerate scales for plane elasticity from the degenerate scale for Laplace's equation for some multiply connected sets: the cases of segments on a line or of arcs of circle with a n-fold symmetry. In these last cases, the exact values of the degenerate scales are obtained when the degenerate scale for the Laplace problem is known.

1. Introduction

The degenerate scales appear when solving single layer boundary integral equations with kernels containing a logarithmic term. This is the case for plane problems related to conduction or elasticity. Among early investigators working on Laplace's equation, we can cite Christiansen (1975); Jaswon (1963). Costabel and Dauge (1996) investigated the case of biharmonic equation. Antiplane elasticity problems are closely related to Laplace's ones and some specific cases have been considered: Joukowski profile Chen (2013), quadrilaterals Chen (2012), regular N-gon domains Kuo et al. (2013b). The case of plane elasticity has been studied in Constanda (1994); Kuhn et al. (1987); Vodička and Mantič (2004). The interest in degenerate scales has increased with the development of Boundary Element Methods, the degenerate scales causing loss of uniqueness and ill conditioning (Chen et al., 2002; Chen and Lin, 2008; Dijkstra and Mattheij, 2007) of the linear system obtained by BEM. Several methods have been developed to get over this problem (Chen et al., 2014, 2015b, 2005; Chen and Lin, 2008; Christiansen, 1982).

The asymptotic behavior of degenerate scales has been investigated for Laplace's equation Corfdir and Bonnet (2013) and for plane elasticity (Chen, 2015; Vodička, 2013). Upper bounds of degenerate scales for plane elasticity have been obtained recently Corfdir and Bonnet (2015). A first work about anisotropic elasticity has been performed by Vodička and Petrík (2015).

The exact values of degenerate scales for Laplace's Boundary Value Problems are known for many cases. They can be obtained by computing the logarithmic capacity of the domain Hayes and Kellner (1972). The name of logarithmic capacity has been given because of "the analogy with the three-dimensional Newtonian case typified by the distribution of electricity on a conductor" Hille (1962). A review of known exact values of logarithmic capacities can be found in Rumely (1989) and examples of application to Laplace's problem in Kuo et al. (2013a). In comparison, the known exact values of degenerate scales for elasticity are scarce. A review of the cases already studied can be found in Corfdir and Bonnet (2015). So, the aim of the present paper is to provide two methods of solution and the exact values of elastic degenerate scales in several application cases. The methods of solution use complex potentials. Indeed, we apply the ideas presented by Muskhelishvili (1953) to solve boundary values problems in plane elasticity using a specific complex representation (see also (England, 2003; Milne-Thomson, 1960; Sokolnikoff, 1956)). This method has been applied to numerous problems, for example the study of stress concentration due to holes and cracks (Savin, 1961; Sneddon and Lowengrub, 1969). A first application of such a method to the degenerate scale problem for elasticity was described in Chen et al. (2009a). These authors use conformal mappings $w(z)$ from the outside of the unit disk to the outside of considered domains. One feature of these conformal mappings $w(z)$ is to behave as z at ∞.

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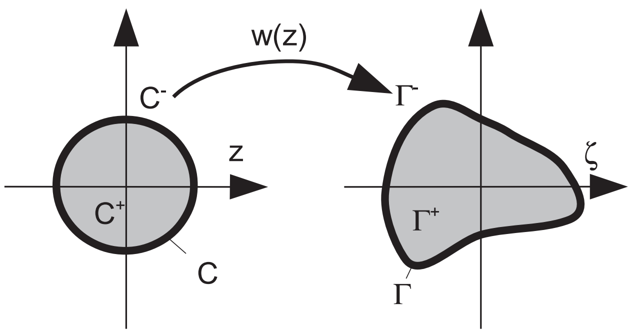

Fig. 1. Notations: conformal mapping w from C^- to Γ^-.

In this paper, a first new method is developed by extending the methodology described in the pioneering paper of Chen et al. (2009a), leading to a more systematic means to obtain the degenerate scales. The computation leads to finite algebraic linear systems. Then, the calculus can be greatly alleviated by the use of symbolic computational softwares.

A second method is developed, using the solution of the Laplace's problem to build the elastic complex potentials for two cases: sets of segments on a line or set of arcs of a circle with a n-fold symmetry axis. We show how to find the exact values of the elasticity degenerate scales in these cases when the exact value of the degenerate scale (or of the logarithmic capacity) for Laplace's problem is known.

2. The null condition at the boundary using conformal mapping and complex potentials

In this paper, the degenerate scale problem for elasticity will be studied for contours that can be described by using conformal mappings (For example, contour Γ in Fig. 1). This section presents the background on conformal mapping and complex potentials that are used to find the degenerate scales. In a first step, the requirements on the conformal mappings that are used to describe different contours are prescribed. Next, the complex potentials that allow us to obtain the solution of plane elasticity problem are recalled, these potentials being also submitted to precise requirements. Degenerate scales correspond to specific contours such that non-null potentials meet a condition of null prescribed displacement at such contours. So, the following step will be to present how this condition of null displacement at the boundary can be prescribed. These non-null potentials will be called in the following "eigenfunctions with 0 eigenvalue".

2.1. Choice of the type of conformal mapping

In the books of Muskhelishvili (1953) and Sokolnikoff (1956) the conformal mappings considered for the study of infinite domains are from the interior of the circle to the exterior of the image of the circle; more recent authors (England, 2003; Milne-Thomson, 1960) find generally more convenient to use the transformation from the outside of the circle to the outside of the image of the circle (Fig. 1). This choice is coherent with the mathematical definition of the class Σ of univalent functions Duren (1983, e.g.); it is also used for the evaluation of the logarithmic capacity and of the degenerate scale for Laplace's equation. As explained thereafter, the contours Γ that will be obtained by the conformal mappings $w(z)$ in Σ are at the degenerate scale for Laplace's equation.

We consider mapping functions defined on C^- and which can be written in the following way:

$$\zeta = Rw(z), \tag{1}$$

where R is a positive real and $\underset{z\to\infty}{\text{limit}}\frac{w(z)}{z} = 1$. Following Duren (1983), the function $w(z)$ is a univalent function of class Σ, i.e. holomorphic on C^- except for a simple pole at infinity with residue 1, and has a series expansion $w(z) = z + \sum_{n=0}^{\infty} m_n z^{-n}$.

In a first step we assume that $0 \notin \Gamma^-$; that is $w \in \Sigma*$ Duren (1983). If it is not the case, it is shown in Section 2.4 that a convenient translation allows us to transform the problem into another one that meets that condition.

It is known that the image of the unit circle by such a mapping has a logarithmic capacity equal to $\ln R$ and is at the degenerate scale for the Laplace's operator if $R = 1$ (Hayes and Kellner, 1972; Kuo et al., 2013a; Yan and Sloan, 1988). So, we will essentially compare the degenerate scales for elasticity to the degenerate scale for Laplace's problem.

2.2. Use of the elastic complex potentials

Following Muskhelishvili (1953), two elastic complex potentials Φ and Ψ can be used to obtain a displacement field solution of the plane elasticity equations in ζ plane. They are written:

$$\Phi(\zeta) = A\ln(\zeta) + C\zeta + \Phi_0(\zeta), \tag{2}$$

$$\Psi(\zeta) = -\bar{A}\ln(\zeta) + C'\zeta + \Psi_0(\zeta) \tag{3}$$

where $\bar{A}$ is the conjugate of A and functions Φ_0 and Ψ_0 are holomorphic in Γ^-. Then, the displacement components u and v are given by:

$$2G(u + iv) = \kappa\Phi - \zeta\overline{\Phi'} - \overline{\Psi} \tag{4}$$

with $\kappa = 3 - 4\nu$ for plane strain problem ($\kappa = (3 - \nu)/(1 + \nu)$ for plane stress problems).

Following Vodička and Mantič (2004), it is also required for the eigenfunctions with zero eigenvalue to meet the condition that the stress field induces a finite resultant force at the boundary and tends to 0 at infinity. A first consequence is that the solutions to the zero eigenvalue problem are such that $C = C' = 0$.

A second consequence of that condition impacts also the value of Φ_0 and Ψ_0 at infinity. We refer to Chen et al. (2009b) and we adopt the following values for the potentials of a concentrated force P at a point t with components P_x and P_y England (2003):

$$\Phi_P = -F\ln(z - t);$$
$$\Psi_P = \kappa\bar{F}\ln(z - t) + F\frac{\bar{t}}{z - t} \text{ with } F = -\frac{P_x + iP_y}{2\pi(\kappa + 1)}. \tag{5}$$

Then, as shown in Chen et al. (2009b), the potentials at infinity can be written in the form (2,3) with $C = C' = 0$ and Φ_0 and Ψ_0 tend to zero when z tends to infinity. This form of Φ_0 and Ψ_0 depends on the choice of the complex potentials Φ_P and Ψ_P related to the concentrated force and is true only for the choice given by (5). It is important to notice that this form of potential for a point force corresponds to an expression of the Green's tensor for plane elasticity that is not the standard Green's tensor (Kelvin's tensor). In addition, modifying the choice of the potential that is used to describe the concentrated forces leads to another value of the degenerate scale. A scaling procedure to convert the degenerate scale for one choice of potential to the degenerate scale for another one is reported in Vodička and Mantič (2004, (3.4)). A consequence is that all degenerate scales obtained in the following must be multiplied by the factor $e^{\frac{1}{2\kappa}}$ to recover the degenerate scales corresponding to the usual Kelvin's tensor.

2.3. The boundary equation for eigenfunctions with 0 eigenvalue

We are looking for non trivial solutions with a null displacement on Γ: then the potentials $\Phi(\zeta)$, $\Psi(\zeta)$ must satisfy a boundary