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Crack identification in non-uniform rods by two frequency data

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ABSTRACT

We consider the inverse problem of identifying a single open crack in a longitudinally vibrating rod having non-uniform smooth profile. Without any a priori assumption on the smallness of the damage and assuming that the rod profile is symmetric with respect to the mid-point of the rod axis, we present a constructive diagnostic algorithm from minimal frequency data. We show that the crack can be uniquely identified, up to a symmetric position, from the first two positive natural frequencies of the rod under free–free end conditions. We also show that the non-uniqueness of the damage location can be removed by using as data the first positive resonant frequency of the free–free rod and the first antiresonant frequency of the driving-point frequency response evaluated at one end of the rod. The results of numerical simulations and of applications of the method to experimental data agree well with the theory.

1. Introduction

Non-uniform vibrating beams are frequently used in engineering applications since they may offer the advantage for a selective distribution of stiffness and mass. This may help in fitting special design requirements and in obtaining optimal dynamical responses.

In spite to the importance of the vibrational behavior of such beams, many studies focussed on uniform beams. The lack of research is all the greater in the case of non-uniform beams with a localized damage, such as a crack, and, particularly, on the inverse problem of identifying the damage from dynamic data. Actually, several researchers considered the identification of a single (open) crack from frequency measurements, but their studies were often restricted to rods with uniform profile, see, among others, the contributions (Springer et al., 1988; Lin and Chang, 2004; Rubio, 2009; Cerri and Vestroni, 2000; Vestroni and Capecchi, 2000). A limited number of researches focussed on crack identification on non-uniform beams. We refer, for example, to the paper Chaudhari and Maiti (2000) for a study of direct and inverse problems for geometrically segmented cracked beams, and to the contributions Adams et al. (1978) and Liang et al. (1992) for resonant frequency-based damage assessment in tapered and piecewise constant cracked beams, respectively.

In this paper we shall concern with the crack identification problem for beams under longitudinal vibrations (rods) by minimal spectral data. One of the first rigorous results on this topic is due to Narkis (1994), who proved that a single small crack in a free–free uniform rod can be uniquely localized (up to a symmetric position) by the first two positive natural frequencies of the longitudinal vibration. Under the assumption that the crack remains open during vibration, the damage was modeled as a translational linearly elastic spring, of stiffness K, located at the cross-section of abscissa s. The stiffness value K can be estimated in terms of the geometry of the cross-section and the mechanical properties of the beam, see Freund and Herrmann (1976). An extended series of experiments confirm the accuracy of the localized flexibility model for cracked rods, particularly for low frequencies, see, among other contributions, Caddemi and Morassi (2013). Narkis's method is based on a perturbation analysis and takes advantage of the fact that the frequency equation for a uniform cracked rod can be written in explicit form.

The identification of a single small crack in a rod with variable profile has been considered in Morassi (2001). Working directly on the weak formulation of the eigenvalue problem, it was shown in Morassi (1993) that the first order change $\delta\lambda_n$ in a generic eigenvalue λ_n (e.g., the resonant frequency squared) is given by

$$\delta\lambda_n = \frac{N_n^2(s)}{K}, \tag{1}$$

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where $N_n(s)$ is the axial force in the nth vibration mode of the undamaged rod, evaluated at the cracked cross-section of abscissa s. It follows that the ratio of the first order changes to two eigenvalues

$$\frac{\delta\lambda_n}{\delta\lambda_m} = \frac{N_n^2(s)}{N_m^2(s)} \tag{2}$$

is known as a function of s only, and it may be possible to find the position of the crack s corresponding to a given (measured) value of $\frac{\delta\lambda_n}{\delta\lambda_m}$. In particular, for a free–free rod with regular profile $a = a(x)$ (e.g., a continuous and continuously differentiable function) and symmetric with respect to the mid-point, it was shown in Morassi (2001) that the knowledge of the first ($m = 1$) and second ($n = 2$) positive eigenfrequencies uniquely determines the position of the crack, up to a symmetric position. Under the same assumptions, the indeterminacy induced by the symmetry of the rod can be removed by using the first resonant frequency of the free–free rod and the first antiresonant frequency of the driving point frequency response function measured at one end of the rod, see Dilena and Morassi (2004).

All the results found in Narkis (1994), Morassi (2001) and Dilena and Morassi (2004) hold under the essential hypothesis that the severity of the crack is small, that is the cracked rod is a perturbation of the undamaged rod. In addition, it should be noticed that the identification algorithm proposed in Morassi (2001) and Dilena and Morassi (2004) is constructive for rods with uniform profile only. The assumption of light damage is reasonable in many practical applications. However, it is not easy to state rigorously when a crack can be considered as small. In fact, even restricting the analysis to the linearized frequency change, Eq. (1) shows that the vibration modes have wavy sensitivity to damage according to the position of the crack, and that the wavy character is more oscillating as the mode order increases. The introduction of an average frequency shift does not simplify the analysis, since it should be clarified how many data must be included in the calculation and how the threshold value corresponding to small damage should be selected. In addition, it is desirable to obtain a unifying general theory of the diagnostic problem capable to include damages ranging from small to large severity.

A first rigorous attempt to solve the inverse problem of detecting a not necessarily small crack in a rod has been presented in Rubio et al. (2015). The authors have shown that the results found in Morassi (2001) and Dilena and Morassi (2004) for a small crack continue to hold even for a crack with any level of severity, provided that the rod is uniform. The proof presented in Rubio et al. (2015) is based on a careful analysis of the solutions of the nonlinear system formed by the frequency equation (which is available in closed form) written for the pair of spectral input data, together with suitable lower and upper bounds derived within the variational theory of eigenvalues.

When a rod has variable profile, no closed form expression for the frequency equation is available and a different approach must be adopted for the identification of damage. This open inverse problem has motivated our research and its solution is the objective of the present study. The main steps of our analysis are as follows.

(i) We introduce an equivalent problem for a vibrating rod with a point mass m at the position s.
(ii) We determine the qualitative behavior of the so called $\lambda - m$ and $\lambda - s$ curves, that is the functions $\lambda_n = \lambda_n(s, \cdot)$ and $\lambda_n = \lambda_n(\cdot, m)$, for fixed s and fixed m, respectively.
(iii) We solve the inverse problem by combining the information contained in the $\lambda - m, \lambda - s$ curves corresponding to the pair of frequency data used in identification.

Let us illustrate with some more details the content of such steps.

Step (i) is based on a transformation of the eigenvalue problem for the cracked rod in an equivalent eigenvalue problem for a rod with a point mass $m = \frac{1}{K}$ located at the cracked cross-section, with suitable coefficients and under proper boundary conditions (see Proposition 2.1 for a precise statement). Therefore, the problem of identifying the crack is transformed in the equivalent problem of determining the location and magnitude of the point mass from a pair of natural frequencies.

Step (ii) is mainly based on the explicit determination of the eigenvalue derivatives with respect to the parameters s and m. The expression of the derivative $\frac{\partial\lambda}{\partial m}$ was used in Morassi and Dilena (2002) in a study of the inverse problem of locating a small point mass in a vibrating rod from natural frequency data. The analysis of the $\lambda - m$ and $\lambda - s$ curves allows to determine the qualitative behavior of the first and second eigenvalue of the free–free rod with respect to the position and intensity of the point mass (see Theorem 5.5 for details). It is exactly at this point that, for technical reasons, we restrict the attention to rods having symmetric profile with respect to the mid-point of the axis.

The results obtained in Step (ii) and the use of suitable general properties of the eigenpairs of the problem, allowed to develop in Step (iii) a reconstruction algorithm for the identification of the point mass (up to a symmetric location) from the first two positive resonant frequencies of the rod. An extended series of numerical simulations on rods having different profile and for various positions and intensities of the crack supported the theoretical results. A selected set of numerical results and some applications to experimental data are presented in Section 9.

The above results can be generalized in a couple of directions. First, the crack identification problem can be formulated and solved in terms of resonant and antiresonant data (see Section 7), thus showing that both the severity and the location of the crack can be uniquely determined by measuring the first positive natural frequency under free–free end conditions and the first antiresonant frequency of the driving frequency response evaluated at one end of the rod. Second, the analysis can be carried out also for symmetric rods in which the linear mass density is not proportional to the axial stiffness function (see Section 8).

The paper is organized as follows. The reduction of the eigenvalue problem for the cracked rod to an equivalent eigenvalue problem for a rod with a point mass is shown in Section 2. Certain basic properties of the eigenvalue problem for the rod with a point mass are listed in Section 3. The first order partial derivatives of an eigenvalue with respect to the point mass and the mass location are determined in Section 4. The behavior of the λ-m and λ-s curves is studied in Section 5. The damage identification algorithm in a free–free rod based on measurements of the first two positive natural frequencies is presented in Section 6. Generalizations to resonant-antiresonant frequency data and to larger classes of rods are illustrated in Section 7 and Section 8, respectively. Section 9 is devoted to numerical and experimental applications of the theory. Proofs of some technical results are collected in Appendix (Section A). We think that the disadvantage of increasing the size of the paper by including the Appendix is by far out-weighed by the fact that Section 3 and, partially, Section 5, together with the Appendix, represent a self-contained approach to the topic.

2. An equivalent eigenvalue problem for a rod with a point mass and main result

Let us consider a longitudinally vibrating free–free straight thin rod of length L. Denote by $\widehat{A} = \widehat{A}(z)$ the area of the transversal