International Journal of Solids and Structures 80 (2016) 465–485

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Diffraction of waves on triangular lattice by a semi-infinite rigid constraint and crack

Basant Lal Sharma*

Department of Mechanical Engineering, Indian Institute of Technology Kanpur, Kanpur, U. P. 208016, India

ARTICLE INFO

Article history:
Received 9 March 2015
Revised 26 August 2015
Available online 10 November 2015

Keywords:
Sommerfeld diffraction
Lattice
Triangular
Crack
Rigid ribbon
Wiener–Hopf

ABSTRACT

An exact solution is provided for a discrete analog of each of the two Sommerfeld diffraction problems using a triangular lattice model, with nearest neighbor interactions, deforming in the anti-plane direction. The discrete Helmholtz equation with time-harmonic data prescribed on semi-infinite row(s) of lattice sites is solved using the discrete Wiener–Hopf method. An asymptotic expression of the exact solution, given in the form of a contour integral, has been obtained in far field using the standard approximation of the diffraction integrals based on the method of stationary phase. In case of the rigid constraint diffraction, the displacement of particles on the semi-infinite row complementing the constrained lattice sites, as well as on the adjacent row, is presented in closed form as a discrete convolution. For the crack diffraction problem, the length of both types of slant bonds on the semi-infinite row complementing the crack, as well as the crack opening displacement, is given in a similar form but in terms of the Fourier coefficients, of the associated Wiener–Hopf kernel, which are not available in closed form. The displacement field associated with the scattered waves at sites far from the defect tip, as well as few sites near the tip, is compared graphically with that of a numerical solution on a finite grid. Both discrete Sommerfeld problems are naturally relevant to their continuous counterparts, involving the traditional Helmholtz equation, that model the diffraction of electromagnetic and acoustic waves by a semi-infinite screen, based on 7-point discretization on a triangular grid.

1. Introduction

Sommerfeld edge diffraction occupies a distinguished status in the mechanics of waves (Born and Wolf, 1999; Felsen and Marcuvitz, 1973; Sommerfeld, 1896; Sommerfeld, 1964). Diffraction of time harmonic waves, by straight-edged semi-infinite half planes, is associated with the ‘twin’ problems in diffraction theory, originally solved by Sommerfeld (1896). Indeed, Sommerfeld’s solutions were the first in the history of such problems and were presented, a century ago, as solutions of the two dimensional Helmholtz equation with boundary conditions on a half plane of either Dirichlet or Neumann. A large class of methods have been applied to solve the Sommerfeld problems and its variants (Chambers, 1954; Copson, 1946; Friedlander, 1946; Lamb, 1907) (also see for example, (Achenbach, 1973; Bouwkamp, 1954; Felsen and Marcuvitz, 1973; Jones, 1964; Noble, 1958)). Out of all known approaches, however, the formulation using an inhomogeneous Wiener–Hopf integral equation has been most popular (Noble, 1958; Wiener and Hopf, 1931). In recent works of the author (Sharma, 2015a; 2015b; 2015c; 2015d), the original Sommerfeld problems have been formulated as *discrete Sommerfeld problems* on a square lattice and are analyzed using the discrete Wiener–Hopf method (Fel’d, 1958; Karp, 1952; Paley and Wiener, 1934; Wiener and Hopf, 1931). As the classical Sommerfeld problems also appear in elastodynamics, in the form of diffraction of elastic shear wave by either a rigid constraint or a crack (Achenbach, 1973; Harris, 2004), the papers (Sharma, 2015a; 2015b; 2015c; 2015d) are motivated by a discrete analog of this particular mechanical example on square lattices. The author’s recent works employ the model, as well as several definitions and notational devices, used by Maradudin (1958); Maradudin et al. (1971); Slepyan (1982); 2002). In this paper, the discrete Sommerfeld problems, as the scattering problems on a grid, have been posed on triangular lattice using the 7-point stencil (Bilbao, 2004; Collatz, 1960; Kantorovich and Krylov, 1958) for the two dimensional Helmholtz equation. The triangular lattice, according to the usage in this paper, is also known as hexagonal lattice (Hahn, 2002) and it has appeared several times in the context of numerical analysis, for example, see Mullen and Belytschko (1982); Zingg and Lomax (1993), as well as mechanics (Dean, 1963; Fineburg and Marder, 1999; Kessler, 1999; Lifshitz and Kosevich, 1966; Makwana and Craster, 2013; Marder, 2004; Marder and Gross, 1995; Marder and Liu, 1993). A well known discrete mechanical model of crack (Slepyan, 2002) has been adapted in problem formulation (see

* fax.: +915122596173; fax: +915122597408.
E-mail address: bls@iitk.ac.in

http://dx.doi.org/10.1016/j.ijsolstr.2015.10.008
0020-7683/

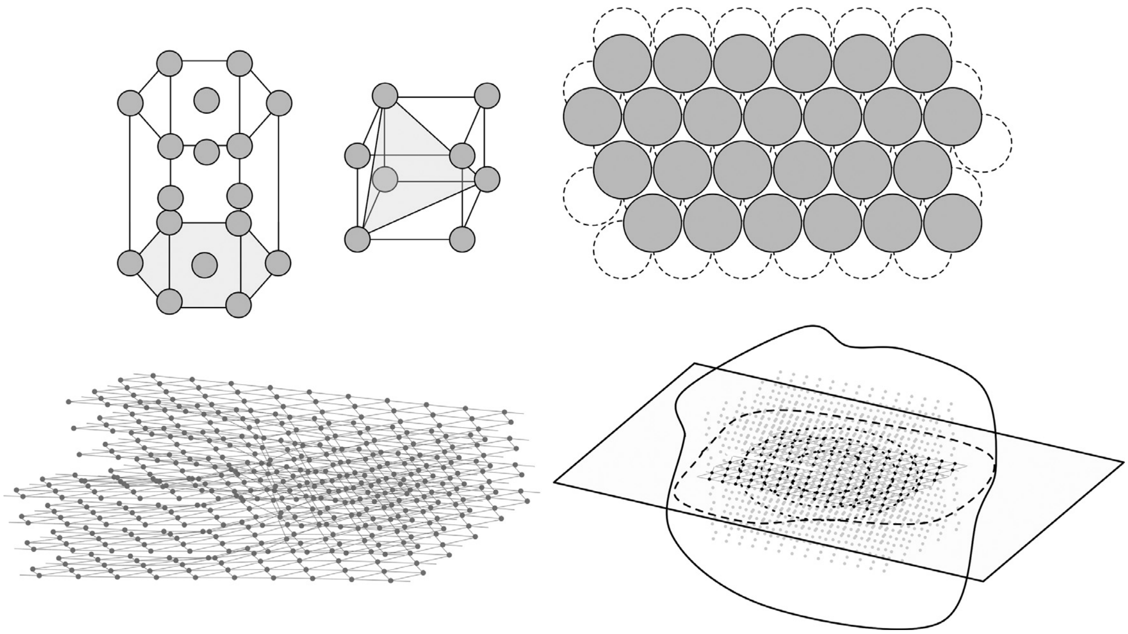

Fig. 1. Triangular lattice $\mathfrak{T}$ as a part of three dimensional purely crystalline body with close packed layers. The anti-plane assumption, with a mode III crack, is schematically shown in both figures appearing below.

also Fineburg and Marder, 1999; Kessler, 1999; Marder and Gross, 1995).

From a point of view of the applications to crystalline materials, the square lattice model is not as realistic as triangular lattice, since close-packed planes occur frequently in several interesting crystals (Born and Huang, 1954; Burke, 1966; Hahn, 2002; Kosevich, 2005) in the form of triangular lattice. Indeed, the fcc and hcp structures are close packed (Conway and Sloane, 1988; Hahn, 2002; IUCr, 1992), where atoms on specific crystallographic planes in the lattice are in the closest possible proximity to one another, i.e., arranged in the form of triangular lattices. For instance, see Fig. 1 for an artwork-based illustration of such three dimensional configuration (the arrangement of second layer in figure on top right also indicates the challenge associated with the assumption of anti-plane motion). The triangular lattice model is also deemed to find some relevance in the investigations into the qualitative properties of out-of-plane (also called anti-plane) deformation of thin layers with the close packed planes parallel to the layer plane. Gronckel (1991); Gong et al (1999); Ohtake and O. Yabuhara (2011) presented some experimental observations of such triangular lattice structures (see also Pierański and Finney, 1979). A survey of the enormous volume of the research works in thin films, monolayers, etc. (Ohring, 2002), even with the restriction of focus on applications relevant to the wave scattering by structural defects, is not an objective of this paper.

Although the discrete formulation has been studied recently by Sharma (2015a); 2015b); 2015c); 2015d) using a square lattice model, its extension to a triangular lattice model is not direct. Similar to the analysis on square lattice (Sharma, 2015a; 2015b), Jones' approach for diffraction problems (Jones, 1952; 1964; Noble, 1958) motivates the solution of the discrete Sommerfeld problem of diffraction involving a semi-infinite crack and rigid constraint on a triangular lattice. As the model is discrete, in place of the continuous Fourier transform, the discrete Fourier transform (Böttcher and Silbermann, 2006; Gohberg and Feldman, 1974; Krein, 1962; Slepyan, 2002) has been employed. The factorization of Wiener–Hopf kernel stated in this paper, a crucial step in any successful application of the Wiener–Hopf technique (Noble, 1958), is explicit in case of rigid constraint, though that is not so in case of crack. Due to certain geometric symmetry, the problem formulation, as well as its solution, for the rigid constraint diffraction is almost identical to the problem, and its solution, for a rigid constraint in square lattice (Sharma, 2015b; 2015d). On the contrary, the problem and solution for crack are quite different due to the presence of two types of bonds in each row of triangular lattice. The peculiar structure in crack problem is manifest in its solution and involves a combination of two 'types' of incident waves. This situation recurs in the case of hexagonal lattice with zigzag constraint (Sharma, 2015e) leading to similarities, between crack diffraction problem on triangular lattice and zigzag constraint diffraction problem on hexagonal lattice (honeycomb).

For convenience, the 'form' of both exact solutions has been intentionally chosen to be the same as its counterpart for diffraction by a semi-infinite crack and a semi-infinite rigid constraint in square lattice, as stated in Sharma (2015a) and Sharma (2015b), respectively. The benefit of this choice appears in the section on asymptotic analysis of the diffracted wave in far field, since it is analogous to that presented in Sharma (2015a) and Sharma (2015b). A rigorous proof of existence and uniqueness of the solution of both problems, constructed in this paper, and on the same lines (based on the theory of Toeplitz operators) as detailed for a square lattice (see Sharma, 2015c and Sharma, 2015d), is present in Sharma (2015f) for the dissipative case.

Aside from the construction of exact solution, the far field approximation of scattered wave field has been also provided in this paper, based on the application of traditional stationary phase approximation (Courant and Hilbert, 1953; Erdélyi, 1955) to the diffraction integral (Born and Wolf, 1999; Felsen and Marcuvitz, 1973) (similar to its application in (Sharma, 2015a)). These expressions, in the form of graphical plots, for some frequencies and a fixed, but arbitrary, angle of incidence, have been compared with a numerical solution of the diffraction problem on a finite grid (Berenger, 1994; Makwana and Craster, 2013; Singer and Turkel, 2004). In addition to this, a closed form expression, obtained by expanding certain functions in power series, for the displacement at lattice sites ahead of, as well as adjacent to, the semi-infinite rigid constraint is provided. Closed form expressions for the bond lengths ahead of the semi-infinite crack as well as the crack opening displacement are also provided in the same way. Again, the results, in the form of graphical plots, for some frequencies and a fixed angle of incidence, provide a