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On the modeling of asymmetric yield functions

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ABSTRACT

A first degree homogeneous yield function is completely determined by its restriction to the unit sphere of the stress space; if, in addition, the function is isotropic and pressure independent, its restriction to the octahedric unit circle, the $\boldsymbol{\pi}$-circle, is periodic and determines uniquely the function. Thus any homogeneous, isotropic and pressure independent yield function can be represented by the Fourier series of its $\boldsymbol{\pi}$-circle restriction. Combinations of isotropic functions and linear transformations can then be used to extend the theory to anisotropic convex functions. The capabilities of this simple, yet quite general methodology are illustrated in the modeling of the yielding properties of AZ31B magnesium alloy.

1. Introduction

In the phenomenological theory of plasticity, the plastic response of metals is described by a yield surface and an associated flow rule. The initial yield surfaces of BCC and FCC alloys, e.g., steel and aluminum, are, within an acceptable approximation, symmetric with respect to the origin of the stress space. By contrast, the initial yield surfaces of HCP-alloys, e.g., magnesium and titanium-alloys, are strongly asymmetric due to twinning at constituent level, e.g., Bilby and Crocker (1965), Christian and Mahajan (1995), Balasubramanian and Anand (2002), Graff et al. (2007), Kouchmeshky and Zabaras (2009); and, even if initially symmetric, a yield surface may become asymmetric due to the residual stresses induced by plastic deformation, e.g., Ortiz and Popov (1983), Zatarin et al. (2004), Barlat et al. (2011).

While the description of symmetric yield surfaces, by employing symmetric yield functions, is a relatively well developed subject, e.g., Bron and Besson (2004), Barlat et al. (2005), Banabic et al. (2005), Vegter and Boogard (2006), Barlat et al. (2007), Soare and Barlat (2010), Huang and Man (2013), a general methodology for developing asymmetric yield functions is still lacking. This may be due to the more sophisticated geometry of an asymmetric surface, which requires a more complex approach. However, the increasing interest, especially from the transportation industry, in lighter materials such as Mg or Ti-alloys, may provide enough motivation for accepting a higher level of complexity in the modeling of a yield surface.

As previous contributions to the problem, we note the work of Liu et al. (1997), where Hill's quadratic was extended via an algebraic combination of orthotropic extensions of the $J_2 := (1/2)|\boldsymbol{\sigma}'|$ and $I_1 := \mathrm{tr}(\sigma)$ invariants of the stress deviator $\boldsymbol{\sigma}'$ and the stress tensor $\boldsymbol{\sigma}$, respectively; a similar approach was adopted later in Cazacu and Barlat (2004), where an algebraic combination of orthotropic extensions of J_2 and $J_3 := \det(\sigma')$ was employed, and further extended to algebraic combinations of general homogeneous polynomials in Soare et al. (2007); also, anisotropic extensions, via linear transformations, of particular isotropic asymmetric functions were proposed by Plunkett et al. (2008) and Yoon et al. (2014).

Noteworthy, the function of Liu et al. (1997) incorporates asymmetric yielding only through pressure-dependence. This approach is subject to debate, for while it is true that pressure-dependence does induce a certain yielding asymmetry, its magnitude is expected, in general, to be small, as evidenced by the experiments of Spitzig and Richmond (2014) on aluminum and steel; more importantly, the nature of this asymmetry is qualitatively distinct from the "intrinsic" yielding asymmetry generated by asymmetric activation conditions at crystal level in HCP-lattices. Indeed, in the absence of microvoids, the pressure-dependence of metals originates in non-Schmid effects which are specific to the lattice arrangement. In particular, Soare and Barlat (2014) have shown that a purely hydrostatic deviation from the Schmid law explains the experiments of Spitzig and Richmond (2014), with aggregates of cubic crystals featuring no plastic dilatancy and the normality rule holding in the deviatoric sections of the macro yield surface; this holds true also for hexagonal lattices with elastic coefficients satisfying (in Voigt notation) $\Delta := (c_{13} + c_{33}) - (c_{12} + c_{11}) = 0$, a relationship representing the

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decoupling of the elastic response into volumetric and deviatoric components. Among the HCP-metals of interest, magnesium has the smallest deviation from this relationship, with $|\Delta| \approx 2$ GPa, and hence a small plastic dilatancy is expected. The recent experiments of Kondori and Benzerga (2014), where a plastic dilatancy of about 4% was reported for AZ31B under homogeneous uniaxial loading, appear to confirm the theory, although the reported dilatancy seems too large to be generated by a pressure effect alone, Barrett et al. (2012).

Thus, in general, both "intrinsic" asymmetry and pressure-dependence should be combined in order to obtain an accurate representation of the overall yielding asymmetry observed in HCP-alloys. This was recently attempted by Yoon et al. (2014) in their modeling of AZ31, where an asymmetric pressure-independent function was combined with a pressure dependent term in order to better capture the yielding asymmetry of AZ31. However, the magnitude of the pressure-dependent term in the cited work is somewhat "arbitrary" and the corresponding flow rule is left unspecified. In the absence of an adequate experimental description of the pressure-dependence of HCP-alloys, in this work pressure-dependence is neglected and the classical normality rule is adopted.

For plastically anisotropic materials, a common approach nowadays consists of combining specific isotropic functions with anisotropic linear transformations of the stress tensor (Barlat et al., 2007; Bron and Besson, 2004; Karafillis and Boyce, 1993; Plunkett et al., 2008; Yoon et al., 2014), the a priori convexity of the resulting anisotropic function for the whole range of its material parameters being the primary motivation of this approach. This method has variants depending on whether the isotropic function is taken to be pressure-dependent or not (i.e., operating on the stress or its deviator) and whether the linear transformation acts itself as a deviatoric "projector". For convenience, here the shorthand term "generator" refers to the isotropic function whose argument is the transformed stress, and is endowed with properties to be described in context. For example, Plunkett et al. (2008) developed an asymmetric, pressure-independent anisotropic yield criterion using a pressure-dependent asymmetric generator. Here we shall use only pressure-independent generators.

The modeling capabilities of the anisotropic functions obtained via the linear transformation approach depend significantly on the generators employed. Here, while retaining the linear transformation approach for generating anisotropic extensions, we aim at developing a general methodology for describing any isotropic and pressure-independent function, symmetric or not. This, in combination with a simple, geometrical method for constructing new isotropic functions, will allow us to exploit the linear transformation approach at its full potential. In a similar context, an early sketch of a general theory of ("plane-isotropic") yield functions, based on trigonometric polynomials, was outlined by Budiansky (1984), although only for plane stress states and symmetric functions. We adopt the same approach, but develop the arguments down to the practical level where applications can be developed with relative ease, in algorithmic manner.

2. Isotropic, pressure-independent asymmetric yield functions for general stress states

We start with a brief review of the natural representation of isotropic functions, the reader being directed to the literature for further details on this classic topic of plasticity, e.g., Hill (1950). Let $\boldsymbol{\sigma}$ denote the stress tensor, $\boldsymbol{e}_i$ its principal directions, and σ_i its principal values. A pressure independent, first order positive homogeneous yield function *f* can be represented as

$$f(\boldsymbol{\sigma}) = f(\boldsymbol{\sigma}') = |\boldsymbol{\sigma}'| f(\boldsymbol{\tau}') \tag{1}$$

where $|\boldsymbol{\sigma}| := \sqrt{\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}}$ is the magnitude (or norm) of a second order tensor, $\boldsymbol{\sigma}' := \boldsymbol{\sigma} - \mathrm{tr}(\boldsymbol{\sigma})/3$ is the deviator of the stress $\boldsymbol{\sigma}$, and $\boldsymbol{\tau}' := \boldsymbol{\sigma}'/|\boldsymbol{\sigma}'|$ is its unit direction. Since the orientation of the principal frame $\{\boldsymbol{e}_i\}$ can be specified by, say, its three Euler angles ψ_i with respect to a material frame, the analytic representation of *f* can be further detailed to

$$f(\boldsymbol{\sigma}) = |\boldsymbol{\sigma}'|\, g(\psi_1, \psi_2, \psi_3, \tau_1', \tau_2', \tau_3')$$

with τ_i' denoting the principal values of the unit direction of the stress deviator, related to those of $\boldsymbol{\sigma}$ by

$$\tau_i' = (\sigma_i - p)/|\sigma'|, \quad \text{where} \quad p := \mathrm{tr}(\sigma)/3$$

In this section we shall be concerned with the representation of isotropic functions, functions that are invariant to any orthogonal transformation of the material axes. Then, by applying three successive rotations to the body, one can bring the material axes along the principal stress directions while leaving the yield function value unchanged; hence:

$$f(\boldsymbol{\sigma}) = |\boldsymbol{\sigma}'|\, g(\tau_1', \tau_2', \tau_3') \tag{2}$$

One can further apply a 90^o rotation of the body about the principal axis $\boldsymbol{e}_1$, leaving again the yield function value unchanged. This rotation switches the principal stresses σ_2 and σ_3. With two other 90^o rotations about $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ available, and the yield function invariant to any combination of them, it follows that the function *g* must be symmetric:

$$g(\tau_1', \tau_2', \tau_3') = g(\tau_2', \tau_1', \tau_3') = g(\tau_1', \tau_3', \tau_2') = \dots \tag{3}$$

Eqs. (2) and (3) give the most general representation of an isotropic scalar function (with one symmetric tensor argument). However, the arguments of the *g*-function are not independent, since they are related by the two constraints $|\boldsymbol{\tau}'| = 1$ and $\tau_1' + \tau_2' + \tau_3' = 0$. With the $\boldsymbol{\pi}$-plane[1] defined by

$$\Pi_o := \{\boldsymbol{\sigma} \,|\, \sigma_1 + \sigma_2 + \sigma_3 = 0\}$$

of unit normal

$$\boldsymbol{n}_o = (\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)/\sqrt{3} \tag{4}$$

the yield function is completely determined by the restriction of the function g to the unit circle of the $\boldsymbol{\pi}$-plane, which will be referred to as the $\boldsymbol{\pi}$-circle. In geometric terms, the yield surface $\overline{\sigma} = f(\boldsymbol{\sigma})$, with $\overline{\sigma}$ denoting a measure of hardening, is a cylinder with generatrices parallel to $\boldsymbol{n}_o$. Let θ denote the polar angle on the $\boldsymbol{\pi}$-circle, measured counterclockwise starting from, say, $\boldsymbol{g}_1$, where $\boldsymbol{g}_i$ denote the projections of $\boldsymbol{e}_i$ onto Π_o. Due to the symmetries in Eq. (3), it is sufficient to consider only the restriction of *g* to the sector $[0, \pi/3]$ of the $\boldsymbol{\pi}$-circle, Fig. 1.

Indeed, symmetry about $\boldsymbol{g}_1$ (actually, about the plane that contains $\boldsymbol{g}_1$ and is orthogonal to Π_o) reduces the range of θ from $[0, 2\pi]$ to $[0, \pi]$; the symmetry about $\boldsymbol{g}_2$ reduces it further to $[0, 2\pi/3]$; finally, the symmetry about $\boldsymbol{g}_3$ reduces the range of θ to $[0, \pi/3]$. Let $h = h(\theta)$ denote the restriction of *g* to the $\boldsymbol{\pi}$-circle. The final analytical expression for the yield function *f*, featuring the isotropy and pressure independence properties is then:

$$f(\boldsymbol{\sigma}) = |\boldsymbol{\sigma}'|\, h(\theta) \tag{5}$$

with $h : \mathbb{R} \longrightarrow \mathbb{R}_+$ uniquely determined by its restriction to the $[0, \pi/3]$ interval as follows: from $[0, \pi/3]$, *h* is extended into $[\pi/3, 2\pi/3]$ by symmetry (corresponding to the symmetry about $\boldsymbol{g}_3$); then *h* is extended to the interval $[0, 2\pi]$ by periodicity, with a $2\pi/3$-period; finally, *h* is extended from $[0, 2\pi]$ to the whole real axis by 2π-periodicity. Hence *h* is an even $2\pi/3$-periodic function. It can be represented, generally, as the cosine series:

$$h(\theta) = a_0/2 + \sum_{k\geq 1} a_k \cos(3k\theta) \tag{6}$$

[1] Since only the principal stresses are of interest here, we may regard the stress tensor as a vector $\boldsymbol{\sigma} = \sigma_i \boldsymbol{e}_i$ of the 3D-space.