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A stress-concentration-formula generating equation for arbitrary shallow surfaces

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ABSTRACT

Analytical understanding of how stress concentrates is invaluable. An equation that generates stress concentration formulas is derived and shown to apply very well to a number of shallow irregularities on surfaces, for the plane stress conditions and to a first-order approximation. Under shallow conditions, for any first-order Hölder-continuous surface function $f(x)$, the derived equation is: $k_t(x) = 1 - 2\mathscr{H}[f'(x)]$, where $\mathscr{H}$ is the Hilbert transform and $f'(x)$ is the spatial derivative of f with respect to the independent variable. It is shown that using this generating equation, well-known traditional results can be easily derived. Also, a number of other stress concentration formulas for various cases are generated. Furthermore, a second-order approximation is introduced, which shows the dependence of k_t on not only the slope but also on the concavity of the surface. The approach used herein can be extended to finding closed-form solutions to other integral equations possessing similar kernels for applications such as the variation of the stress intensity factor due to an arbitrary crack front profile (work in progress).

1. Introduction

Analytical understanding of how stress concentrates on surfaces possessing an arbitrary topography is of crucial importance in many applications. For various reasons, it is highly desirable to know if and how a material's parcel would become the "weakest link" during the material's service lifetime. Additionally, analytical solutions of how the stress concentrates provide a more effective route to approach certain multi-physics problems by coupling those solutions with, for example, the Navier–Stokes, Fick's Law of diffusion, and free-energy Gibbs equations, just to name a few. Furthermore, stress-driven-reaction analyses provide insight about the stability of surfaces, but it is evident that one limitation is the unavailability of stress-concentration solutions for general configurations of surfaces. Much of the advancement on the foregoing topics has relied on the use of surface-stress-concentration distributions for the cases of sinusoidal surfaces (Gao, 1991a; Liang and Suo, 2001), cycloid rough surfaces (Chiu and Gao, 1993; Li et al., 1993), and some single-notch cases (Yu, 2005). In this document, a novel, 2-dimensional, general equation that generates stress concentration formulas for a wide variety of shallow-surface configurations is presented. Additional impetus for this work follows.

It has been established that surfaces are not inert to their surroundings, and that the latter interplay with the elastic and plastic flows of material at the boundaries. At such inequilibrium conditions, surfaces undergo transformations which, in turn, could be catastrophic for films, interfaces, coatings, etc. For example, electronic and micro-electromechanical devices can be significantly impacted by the foregoing conditions, mostly due to the presence of large internal stresses introduced during the manufacturing processes. It has been reported that stresses in the 1-GPa order of magnitude can be present in the thin films that comprise integrated circuits and magnetic disks (Nix, 1989). A slight magnification of such high stress levels, due to surface deformation, would definitely increase the likelihood of failure. It has been found (Medina and Hinderliter, 2014) that even slightly random rough surfaces (with heights normally distributed) can most likely magnify the bulk stress by a factor of 1.6. For the case of slightly undulating surfaces, this magnifying factor can range from 2 to 3[1] (Gao, 1991b). An analytical description of the distribution ofconcentrated stress for an arbitrary surface would provide safety envelopes for surface topographies found experimentally. Furthermore, stress-driven surface evolution has been recognized to be an important process in the behavior of heteroepitaxial films (Gao, 1994). For specific surface morphologies, advancement on stress-driven reactions has been accomplished

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[1] Although it is shown in this document, in Section 4, via a second-order approximation that these numbers are a few percents lower.

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by coupling analytical solutions of surface stress distribution to the total energy equation.

Ever since the development of the first analytical solution for the stress concentration due to an elliptical hole in a plate provided by Inglis (Inglis, 1913), an immense number of solutions have been developed (Pearson, 1997; Medina et al., 2014; Neuber, 1958). Additionally, some attempts have been carried out to consolidate stress concentration formulas and factors, mostly via empirical methods (Pearson, 1997; Medina et al., 2014; Neuber, 1958; Noda and Takase, 1999). Although the foregoing referenced work is commendable, it is mostly focused on stress-concentration factors and not on the stress distribution along the surfaces of the geometries analyzed. In the present work, a stress-concentra tion-formula-generating equation is derived and shown to be applicable to a wide number of shallow surfaces under tension or pure bending, at least. Under shallow[2] conditions, for any function[3], $f(x)$ descriptive of a 2-dimensional surface, the stress concentration along the profile is given by:

$$k_t(x) = 1 - 2\mathscr{H}(f'(x)) \tag{1}$$

where $\mathscr{H}$ stands for the general Hilbert transform, and prime represents the first derivative with respect to the given independent variable. It will be shown that using this generating equation, well-known traditional results can be easily obtained.

This paper is organized in the following manner. First, the main result, a stress-concentration-formula-generating equation (SCFGE) is derived in a fashion similar to that used by the author in a previous work (Medina and Hinderliter, 2014; Medina and Hinderliter, 2012). Next, the robustness of the SCFGE is shown via derivation of previously developed and accepted results such as semi-elliptical (and semi-circular) single notch, parabolic single notch, sinusoidal surfaces, etc. Following, the stress concentration is found for other interesting profiles, and a table containing results is provided. Next, a second-order approximation is introduced and applied to the undulating surface case. Finally, this document is completed with a discussion and conclusion section.

2. Derivation

For the first part of our derivation, we employ a first-order perturbation method. A similar approach was previously used by Banichuck (1970) and Goldstein and Salganik (1970, 1974), later by Cotterell and Rice (1980), and Rice (1985). In the foregoing references, the perturbation approach was applied to the problem of the stress intensity variation due to a crack front that deviates from flatness due to local irregularities in materials (e.g. grain boundaries). Later, this method has been used by Gao (1991b) and later by the author (Medina and Hinderliter, 2014) to develop, respectively, stress concentration factor formulas for undulating surfaces and random rough surfaces. Other related problems, such as the study of the elastic fields at the interfaces between dissimilar materials (Gao, 1991a; Grekov, 2011), have been studied using the perturbation method, as well.

Consider the profile from Fig. 1, which is a slight perturbation from an originally flat surface. Profile $f(x)$ is a continuous real function that satisfies the Hölder condition[4] within its domain. Fig. 2 shows some typical topographies described by f. Being f a perturbation from a previously flat configuration, an important constraint on f is that it must be of small magnitude. Assume that our surface $f(x)$ is being subjected to a remote tensional load σ far enough from the surface features that Saint–Venant's principle applies (Saint-Venant, 1855). This model can be applied, as well, to a thick rectangular beam with a slight rough surface subjected to pure bending[5], since a sufficient number of layers of fibers on and near one of surfaces can be assumed to be subjected to a tensional load. Following (Medina and Hinderliter, 2014), we can define the stress and displacements for the rough surface (T_{ij} and u_i) in terms of the reference state values (T^r_{ij} and u^r_i) and the perturbed elements (δT_{ij} and δu_i), as:

$$\begin{aligned} T_{ij} &= T^r_{ij} + \delta T_{ij} \\ u_i &= u^r_i + \delta u_i \end{aligned} \tag{2}$$

Consider a concentrated point force F_p (see Fig. 1) acting at an arbitrary location (x, y). The purpose of using a point force is only to utilize the well-known solution for the surface stress Green's function, and the magnitude of F_p can be shown to be irrelevant. Using a similar reasoning as used by others previously (Banichuck, 1970; Goldstein and Salganik, 1970; Goldstein and Salganik, 1974; Rice, 1985) and developed in both (Gao, 1991b and Medina and Hinderliter, 2014), it is shown that perturbation contributions for the stress and displacements are given as:

$$\begin{aligned} \delta T_{ij}(x,y) &= -\sigma \int_{-\infty}^{\infty} \widehat{G}_{ij}\delta f(\chi)d\chi \\ \delta u_i(x,y) &= -\tfrac{\sigma}{E'} \int_{-\infty}^{\infty} G^i_{xx}\delta f(\chi)d\chi \end{aligned} \tag{3}$$

which is for the particular case when there is no concentrated force and thus the material is only exposed to the bulk stress; that is, $T^{F_p=0}_{xx} = T^r_{xx} = \sigma$.

In Eq. (3) E' is the consolidated Young modulus of elasticity[6]; And $\widehat{G}_{ij}$ is a tensor representing a kernel function which can be derived from the Green's function, G^i_{xx}, in the following manner:

$$\widehat{G}_{ij} = \frac{1}{E'}\left[\mu\left(\frac{\partial G^i_{xx}}{\partial x_j} + \frac{\partial G^j_{xx}}{\partial x_i}\right) + \frac{2\mu\nu\delta_{ij}}{1-2\nu}\frac{\partial G^k_{kk}}{\partial x_k}\right] \tag{4}$$

where, in this case, δ_{ij} is the Kronecker delta, and μ is the shear modulus.

The Green's function G^i_{xx} can be obtained by differentiating the strain energy density with respect to the point force F_p (for details see (Medina and Hinderliter, 2014)) and found to be:

$$G^i_{xx}(\chi; x, y) = \frac{\partial T_{xx}}{\partial F^i_p} \tag{5}$$

where the stress Green's function in Eq. (5) is understood as the surface stress at χ due to a unit point force F_p in the i-direction applied at the point source (x, y). Therefore, the Green's function above, Eq. (5), can be used to find the kernel of Eq. (4), which in turned can be used to find the contribution of the stress due to the perturbation. Finally, this perturbation part of stress can be used to find the stress increased from the bulk stress, according to part (a) of Eq. (2).

Extracting the other *Green's functions* G^i_{ij} from the point force solutions for a half-plane from (Green and Zerna, 1968), it can be shown that Eq. (2)(a) can be expressed as:

$$\delta T_{xx}(x,y) = \sigma \int_{-\infty}^{\infty} \frac{2}{\pi(\chi - x)^2}\delta f(\chi)d\chi \tag{6}$$

[2] Shallow means that the heights or depths of hillocks or hollows on the surface are small in comparison to their widths. For example, in the case of a sinusoidal surface, the amplitude is much smaller than the wavelength; or in the case of a single notch, the depth of the notch is much smaller than its width.

[3] Actually, as it will be shown, the requirements to be imposed on f do not limit it from any practical applications.

[4] This is a very important characteristic of our solution, and it is completely related to a property of the Hilbert transform to be discussed later. Most functions found in practical applications meet this criterion. Moreover, for the particular case of surfaces without cracks, this criterion is obviously always met.

[5] Either a four-point bending or a three-point bending with negligible shear.

[6] $E' = \begin{cases} E, & \text{for plane stress} \\ \frac{E}{1-\nu^2}, & \text{for plane strain}, (\nu = \textit{Poisson's ratio}) \end{cases}$