International Journal of Solids and Structures 69–70 (2015) 131–151

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Localization study of a regularized variational damage model

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ARTICLE INFO

Article history:
Received 17 December 2014
Received in revised form 22 May 2015
Available online 16 June 2015

Keywords:
Damage
Failure
Dissipation
Non-locality
Variational approach
Bifurcation

ABSTRACT

The paper presents a detailed analysis and extended formulation of a rate-independent regularized damage model proposed by Mielke and Roubíček (2006). Localization properties are studied in the context of a simple one-dimensional problem, but the results reveal the fundamental features of the basic model and of its modified versions. The initial bifurcation from a uniform solution is described analytically while the complete failure process is studied numerically. Modifications of the regularizing term and of the dissipation distance are introduced and their effect on the global response is investigated. It is shown that, with a proper combination of model parameters, a realistic shape of the load–displacement diagram can be achieved and pathological effects such as extremely brittle response or expansion of the damage zone accompanied by stress locking can be eliminated.

1. Introduction

In engineering mechanics, *damage* is understood as a load-induced evolution of microstructural defects, resulting in a reduced macroscopic material integrity. Phenomenological constitutive models of damage characterize such irreversible phenomena by an *internal* damage variable (Kachanov, 1958), which is closely related to the reduction of the secant modulus of elasticity. Since the seminal contribution of Bažant (1976), it has been well-understood that such a description within the framework of local (i.e. scale-free) continuum mechanics leads to an ill-posed problem, resulting in localization of damage growth into an arbitrarily small region. As a remedy to this pathology, a plethora of non-local rate-independent continuum theories, based on integral, explicit and implicit gradient approaches, have been proposed to introduce an *internal length scale* into the description, see e.g. Bažant and Jirásek (2002) for a representative overview. Despite a significant increase in objectivity offered by the enhanced continuum theories, the non-local damage formulations often suffer from the fact that the non-local variables are introduced into the model in an ad hoc fashion, thus violating basic constraints of thermodynamics. In addition, since the principle of local action is no longer valid, such inconsistencies are rather difficult to detect, especially in the multi-dimensional setting, e.g. Simone et al. (2004). Fortunately, as demonstrated by Jirásek (1998) and confirmed by a number of independent studies, e.g. (Peerlings et al., 2001; Jirásek and Rolshoven, 2003, 2009a,b; Di Luzio and Bažant, 2005; Engelen et al., 2006), a simple one-dimensional study of the localization behavior can serve as a convenient "filter" test, allowing to pinpoint various inconsistencies in the formulation of a constitutive model. The same point of view has recently been adopted by Pham et al. (2011) and Pham and Marigo (2013), who investigated various aspects of the response a wide class of energy-based gradient damage models under displacement-controlled uniaxial tension. These works build on a variational framework for local and gradient-based models developed by Pham and Marigo (2010a,b), in which evolution follows from physically sound principles of stability, energy balance, and irreversibility, expressed using a single energy functional. In particular, Pham et al. (2011) concentrates on the stability of homogeneous solutions, while in the follow-up work (Pham and Marigo, 2013) the authors study in detail the behavior inside the damaged zone and its implications for the structural response. In both cases, the material constitutive law is incorporated in the model indirectly by means of parametrized energy families with parameters adjusted to reproduce the local stress–strain response of the material under investigation. The purpose of our paper is to complement these developments with detailed localization studies for gradient damage models based on the commonly used local stress–strain diagrams. To this purpose, we start from the discussion of an elementary elastic–brittle model regularized by the gradient of damage in the spirit of Frémond and Nedjar (1996); see Section 2. Our description builds on a general framework established by Mielke and co-workers, see e.g. Mielke (2005) for an overview, developed to study the

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http://dx.doi.org/10.1016/j.ijsolstr.2015.06.001
0020-7683/

evolution of general irreversible rate-independent systems, which has been applied to rigorous analysis of gradient damage models and their numerical approximation (Mielke and Roubíček, 2006; Bouchitté et al., 2009; Mielke et al., 2010; Thomas and Mielke, 2010; Mielke, 2011a). The variational formulation presented in Section 2 is thus based on a stored energy functional, quantifying the reversibly stored energy, and a dissipation distance accounting for the irreversible changes. The stored energy is further decomposed into a standard (elastic) part and a regularizing part which introduces a characteristic length into the formulation.

Section 3 presents a study of the localization behavior of the model, utilizing arguments of local incremental energy minimization. Following our recent developments (Jirásek et al., 2013), in Section 3.1 we show that the damage profiles during the damage evolution must be continuously differentiable in space, thereby justifying the assumption made by Pham and Marigo (2013, Remark 2), and derive the continuity conditions at the interface between elastic and damaging regions, as well the governing equations to be satisfied in the region experiencing damage. These conditions are employed in Section 3.2 to characterize the elastic response, in Section 3.3 to obtain an analytical solution to the damage profile at the onset of damage, and in Section 3.4 to study the response at later stages by means of a numerical procedure described in Appendix A. It turns out that the model is regularized in the sense that the energy dissipation is finite, but the global response is extremely brittle, especially at late stages of the failure process. This motivates the search for modifications which could lead to load–displacement diagrams that better correspond to the actual behavior of quasibrittle materials.

In Section 4, the elastic–brittle core of the model is replaced by linear or exponential softening via modifications of the dissipation distance. In Section 5, an elastic–brittle model with the regularizing part of the stored energy dependent on the gradient of a modified internal variable is developed and its alternative interpretation in terms of a variable characteristic length is suggested. Finally, Section 6 combines the linear or exponential softening with variable characteristic length.

2. Variational formulation of elastic–brittle model

We consider a prismatic bar of initial length L, subjected to displacement-controlled uniaxial tensile loading. In the sequel, the bar will be represented by the interval $\Omega = (-L/2; L/2)$, with boundary $\Gamma = \{-L/2, L/2\}$ (consisting of two points) subjected to the Dirichlet loading $u_D(t) : \Gamma \to \mathbb{R}$, where $t \in [0; T]$ denotes the (pseudo-) time; see Fig. 1. For the sake of simplicity, we denote by e the bar elongation (change of length), i.e., we set $e(t) = u_D(t, L/2) - u_D(t, -L/2)$ in what follows.

Following the standard thermodynamic approach to constitutive modeling, summarized e.g. in Chapter 25 of Jirásek and Bažant (2002), a state of the system is described using *admissible* displacement and damage fields $\widehat{u} : \Omega \to \mathbb{R}$ and $\widehat{\omega} : \Omega \to \mathbb{R}$. Formally, we write

$$\widehat{u} \in \mathbb{K}(t) = \left\{ \widehat{u} \in W^{1,2}(\Omega), \widehat{u}(x)|_\Gamma = u_D(t) \right\} \tag{1}$$

$$\widehat{\omega} \in \mathbb{Z} = \left\{ \widehat{\omega} \in W^{1,2}(\Omega), 0 \leqslant \widehat{\omega}(x) \leqslant 1 \text{ in } \Omega \right\} \tag{2}$$

where $\mathbb{K}(t)$ denotes the set of kinematically admissible displacements at time t, $\mathbb{Z}$ stands for the set of admissible damage fields, and $W^{1,2}(\Omega)$ is the Sobolev space of functions with square-integrable distributional derivatives; see e.g. Rektorys (1982).

Within the adopted variational framework (Mielke and Roubíček, 2006), the constitutive description of the damage model is based on

1. The *stored energy* functional
$$\mathcal{E}(\widehat{u}, \widehat{\omega}) = \mathcal{E}_{std}(\widehat{u}, \widehat{\omega}) + \mathcal{E}_{reg}(\widehat{\omega}) \tag{3}$$
with the *standard* part $\mathcal{E}_{std} : W^{1,2}(\Omega) \times \mathbb{Z} \to \mathbb{R}$ and the *regularizing* part $\mathcal{E}_{reg} : \mathbb{Z} \to \mathbb{R}$ respectively defined as
$$\mathcal{E}_{std}(\widehat{u}, \widehat{\omega}) = \frac{1}{2} \int_\Omega (1 - \widehat{\omega}(x)) E \widehat{u}'^2(x) \, dx \tag{4}$$
$$\mathcal{E}_{reg}(\widehat{\omega}) = \frac{1}{2} \int_\Omega g_{f0} \ell_0^2 \widehat{\omega}'^2(x) \, dx \tag{5}$$
where $\widehat{u}'$ corresponds to an admissible strain field $\widehat{\varepsilon}$,
2. The *dissipation distance* $\mathcal{D} : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R} \cup \{+\infty\}$
$$\mathcal{D}(\widehat{\omega}_1, \widehat{\omega}_2) = \begin{cases} \int_\Omega g_{f0}(\widehat{\omega}_2(x) - \widehat{\omega}_1(x)) \, dx & \text{if } \widehat{\omega}_2 \geqslant \widehat{\omega}_1 \text{ in } \Omega \\ +\infty & \text{otherwise} \end{cases} \tag{6}$$

Physically, $\mathcal{E}$ represents the energy reversibly stored in the system and $\mathcal{D}$ is the energy dissipated by changing the damage field from $\widehat{\omega}_1$ to $\widehat{\omega}_2$. The reversibly stored energy consists of the standard part $\mathcal{E}_{std}$ and the regularizing part $\mathcal{E}_{reg}$; the latter depends on the damage gradient and acts as a localization limiter. Note that $\mathcal{E}_{reg}$ vanishes for uniform damage states. In Eqs. (4)–(6), E [Pa] denotes the Young modulus, g_{f0} [Jm^{-3}] is the amount of energy needed to disintegrate a unit volume of the material, and ℓ_0 [m] is a characteristic material length, which reflects the size and spacing of dominant heterogeneities in the microstructure. Later it will become clear that the "$+\infty$" term appearing in (6) enforces irreversibility of damage evolution, i.e., ensures that the damage variable cannot decrease in time.

Now, given the Dirichlet loading u_D, functionals $\mathcal{E}$ and $\mathcal{D}$ and initial data $\bar{u}_0 \in \mathbb{K}(0)$ and $\bar{\omega}_0 \in \mathbb{Z}$, the *energetic solution* of the damage problem is provided by time-dependent fields $u(t) \in \mathbb{K}(t)$ and $\omega(t) \in \mathbb{Z}$ satisfying (Mielke, 2005):

Global stability: for all $t \in [0; T]$, $\widehat{u} \in \mathbb{K}(t)$ and $\widehat{\omega} \in \mathbb{Z}$
$$\mathcal{E}(u(t), \omega(t)) \leqslant \mathcal{E}(\widehat{u}, \widehat{\omega}) + \mathcal{D}(\omega(t), \widehat{\omega}) \tag{7}$$
Energy equality: for all $t \in [0; T]$
$$\mathcal{E}(u(t), \omega(t)) + \mathrm{Var}_\mathcal{D}(\omega, [0; t]) = \mathcal{E}(u(0), \omega(0)) + \int_0^t \int_\Gamma R(s) \dot{u}_D(s) \, d\Gamma \, ds \tag{8}$$

where

$$\mathrm{Var}_\mathcal{D}(\omega, [0; t]) = \sup \sum_{j=1}^{J} \mathcal{D}(\omega(t_{j-1}), \omega(t_j))$$

is the energy dissipated during the time interval $[0; t]$ (with the supremum taken over all partitions of $[0; t]$ in the form

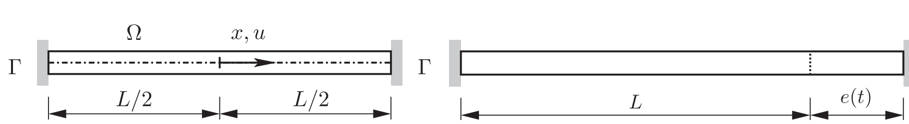

Fig. 1. Bar under uniaxial displacement-controlled tension.