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Deformation of chiral elastic cylinders composed of two materials

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ABSTRACT

The behavior of chiral elastic materials is of interest to the fields of auxetic materials, carbon nanotubes, honeycomb structures and mechanics of bone. The chiral effects cannot be described within classical elasticity. In this paper we study the deformation of isotropic chiral solids by using the theory of Cosserat elasticity. We investigate the behavior of a bar composed by two different materials. The intended applications of the solution are to bone implants and various compound cylinders. The bar is reinforced by a longitudinal rod and is subjected to extension, bending and torsion. It is shown that the compression of the composed cylinder, in contrast with the result predicted by the theory of achiral materials, is accompanied by torsion and bending. The method is used to investigate the case of a circular cylinder reinforced by a circular rod. In this case the compression of the bar produces only a twist around its axis.

1. Introduction

The behavior of chiral materials is of interest for the investigation of carbon nanotubes (Chandraseker and Mukherjee, 2006; Guz et al., 2007; Chandraseker et al., 2009), auxetic materials (Lakes, 1991; Lakes, 1998; Prall and Lakes, 1997; Spadoni and Ruzzene, 2012) and bones (Lakes et al., 1983; Park and Lakes, 1986). The deformation of chiral elastic solids cannot be described by means of the classical elasticity (Lakes, 2001). In many papers the behavior of chiral materials is studied by using the theory of Cosserat elasticity (Park and Lakes, 1986; Lakes, 1987; Lakes, 2001; Healey, 2002; Chandraseker et al., 2009; Ieşan, 2010; Eremeyev and Pietraszkiewicz, 2012). The Cosserat theory studies continua with oriented particles which have the six degree of freedom of a rigid body (Truesdell and Toupin, 1960; Eringen, 1999; Dyszlewicz, 2004,Epstein, 2010). In chiral materials qualitative new phenomena are predicted. Lakes and Benedict (1982) studied the deformation of an elastic cylinder of circular cross-section, made of an isotropic and homogeneous chiral Cosserat material. The cylinder is stretched by an axial force and the lateral surface is free of tractions. It is shown that the rod is predicted to undergo torsional deformation when is subjected to tensile load.

The linear theory of Cosserat elasticity has been used in many papers to describe the mechanical behavior of bones (Yang and Lakes, 1982; Lakes et al., 1983; Park and Lakes, 1986; Fatemi et al., 2002; Fatemi et al., 2003). Lakes, 1987 stated that "Human bone, a natural fibrous composite, displays size effects in torsion and bending which are consistent with Cosserat elasticity rather than classical elasticity". Yang and Lakes (1982) presented some experimental observations on the elastic properties of human compact bone.In recent years the mechanical behavior of bones has been the object of intensive research (Cowin, 2001; Cowin and Doty, 2007). Many papers have been devoted to the study of bone implants (Hanumantharaju and Shivanand, 2009; Thielen et al., 2009). We can assume that the bone and the implant form a body, B, which can be modeled as a continuum composed of different materials. In the last decades the theory of elasticity has been used extensively to model biological tissues. It has been recognized that biological tissues can be modeled under certain conditions as elastic and that understanding this elastic response is a useful preliminary step to studying the more complicated viscoelastic behavior (Wilberg and Walton, 2002). We assume that B is composed of two different elastic materials.

In this paper we study the torsion, bending and extension of a bar which is composed of two materials, welded together along the surface of separation. The intended applications of the solution are to femur bone implants and various compound cylinders. We assume that the bar is composed of two homogenous and isotropic chiral Cosserat elastic materials. The paper is structured as follows. First, we present the basic equations of isotropic chiral Cosserat elastic solids and formulate the problem of extension, bending and torsion for a reinforced bar. Then, we define the generalized plane strain problem associated to the composed body, and introduce four special auxiliary plane problems. In the following section we present the solution of the problem of extension, bending and

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torsion. It is shown that, in contrast with the result predicted by the theory of achiral materials, the torsion of the composed chiral bar is accompanied by extension and bending. Finally, the method is used to investigate the deformation of a reinforced circular cylinder. In this case the compression of the bar produces only a twist around its axis. In the context of the classical elasticity, the deformation of heterogeneous elastic bodies has been studied in various works (Muskhelishvili, 1953; Fichera, 1972; Kupradze et al., 1979).

2. Statement of the problem

In this section we present the basic equations of the equilibrium theory of isotropic chiral Cosserat elastic bodies and the formulation of the problem. Throughout this paper B denotes a right cylinder of length h with the cross-section Σ and lateral boundary Π. We call ∂B the boundary of B, and designate by n_i the components of the outward unit normal of ∂B. Throughout this paper a rectangular cartesian coordinate system Ox_k, $(k = 1,2,3)$, is used. The rectangular cartesian coordinate frame is chosen such that the x_3-axis is parallel to the generators of B and the x_1Ox_2 plane contains one of terminal cross-sections. We denote by Σ_1 and Σ_2, respectively, the cross-section located at $x_3 = 0$ and $x_3 = h$. We assume that the generic cross-section Σ is a regular region. Let L be the boundary of the region Σ_1. We shall employ the usual summation and the differentiation conventions: Greek subscripts are understood to range over the integers $(1,2)$, whereas Latin subscripts (unless otherwise specified) to the range $(1,2,3)$; summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding cartesian coordinate.

Throughout this paper we consider the linear theory of homogeneous and isotropic chiral Cosserat elastic bodies. Let u_j be the displacement vector, and let φ_j be the microrotation vector. The strain measures are given by

$$e_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k, \quad \kappa_{ij} = \varphi_{j,i}, \tag{1}$$

where ε_{ijk} is the alternating symbol. We denote by t_{ij} the stress tensor and by m_{ij} the couple stress tensor. The constitutive equations of homogeneous and isotropic chiral Cosserat elastic materials are (Eringen, 1999; Lakes, 2001)

$$\begin{aligned} t_{ij} &= \lambda e_{rr}\delta_{ij} + (\mu+\kappa)e_{ij} + \mu e_{ji} + C_1\kappa_{rr}\delta_{ij} + C_2\kappa_{ji} + C_3\kappa_{ij}, \\ m_{ij} &= \alpha\kappa_{rr}\delta_{ij} + \beta\kappa_{ji} + \gamma\kappa_{ij} + C_1 e_{rr}\delta_{ij} + C_2 e_{ji} + C_3 e_{ij}, \end{aligned} \tag{2}$$

where δ_{ij} is the Kronecker delta, and λ, μ, κ, α, β, γ and C_k are constitutive constants. The surface force and the surface moment acting at a regular point of ∂B are defined by

$$t_i = t_{ji}n_j, \quad m_i = m_{ji}n_j,$$

respectively. The equilibrium equations, in the absence of body loads, can be written in the form

$$t_{ji,j} = 0, \quad m_{ji,j} + \varepsilon_{irs}t_{rs} = 0. \tag{3}$$

We assume that the considered cylinder is free of lateral loads. Thus, we have the following conditions

$$t_{\alpha i}n_\alpha = 0, \quad m_{\alpha i}n_\alpha = 0 \quad \text{on } \Pi. \tag{4}$$

We suppose that the cylinder B is subjected to extension, bending and torsion. Let $\boldsymbol{R} = (0,0,R_3)$ and $\boldsymbol{M} = (M_1,M_2,M_3)$ be prescribed vectors representing the resultant force and the resultant moment about O of the tractions acting on Σ_1. On Σ_2 there are tractions applied so as to satisfy the equilibrium conditions of the body. Consequently, for $x_3 = 0$ we have the conditions

$$\int_{\Sigma_1} t_{3\alpha}da = 0, \tag{5}$$

$$\int_{\Sigma_1} t_{33}da = -R_3, \tag{6}$$

$$\int_{\Sigma_1} (x_\alpha t_{33} - \varepsilon_{3\alpha\beta}m_{3\beta})da = \varepsilon_{\alpha\beta 3}M_\beta, \tag{7}$$

$$\int_{\Sigma_1} (\varepsilon_{\alpha\beta 3}x_\alpha t_{3\beta} + m_{33})da = -M_3. \tag{8}$$

Let Γ be a closed curve contained in Σ_1, which is the boundary of a regular domain A_2 contained in Σ_1. We assume that L and Γ have no common points. We denote by A_1 the regular domain bounded by the curves L and Γ. Let B_ρ be the cylinder defined by $B_\rho = \{(x_1,x_2,x_3) : (x_1,x_2) \in A_\rho, 0 < x_3 < h\}$, $(\rho = 1,2)$. We suppose that B_ρ is occupied by an isotropic chiral Cosserat elastic material with the constitutive coefficients $\lambda^{(\rho)}$, $\mu^{(\rho)}$, $\kappa^{(\rho)}$, $\alpha^{(\rho)}$, $\beta^{(\rho)}$, $\gamma^{(\rho)}$ and $C_k^{(\rho)}$, $(\rho = 1,2)$. We denote by S the surface of separation of the two materials, $S = \{(x_1,x_2,x_3) : (x_1,x_2) \in \Gamma, \ 0 \leqslant x_3 \leqslant h\}$. We can consider that the cylinder B is composed of two different materials which are welded together along S (Fig. 1).

Assume that in the course of deformation, there is no separation of material along S. The displacement vector, the microrotation vector, the surface force and the surface moment must be continuous in passing from one medium to another. Accordingly, we have the conditions

$$[u_i]_1 = [u_i]_2, \quad [\varphi_i]_1 = [\varphi_i]_2, \quad [t_{\alpha i}]_1 n_\alpha^0 = [t_{\alpha i}]_2 n_\alpha^0, \quad [m_{\alpha i}]_1 n_\alpha^0 = [m_{\alpha i}]_2 n_\alpha^0, \tag{9}$$

on S, where we have indicated that the expressions in brackets are calculated for the domains B_1 and B_2, respectively. Here, n_α^0 are the direction cosines of the vector normal to Γ, outward to A_1.

We assume that the elastic potential corresponding to the material that occupies B_ρ is a positive definite quadratic form in the strain measures. The restrictions imposed by this assumption on the constitutive coefficients are presented in various papers (Lakes and Benedict, 1982; Dyszlewicz, 2004).

The problem consists in finding the functions u_i and φ_i which satisfy the Eqs. (1)–(3) on B_ρ, $(\rho = 1,2)$, the conditions (4) on Π, the conditions (5)–(8) on the end located at $x_3 = 0$, and the conditions (9) on S, when the constants R_3 and M_k are prescribed. If $R_3 = 0$ and $M_\alpha = 0$, then we obtain the torsion problem.

3. Auxiliary generalized plane strain problems

In this section we introduce the generalized plane strain associated to the composed cylinder B. We suppose now that a body force $f_j^{(\rho)}$ and a body couple $g_j^{(\rho)}$ are prescribed on B_ρ. We consider that on the lateral surface Π there are prescribed the surface force $\tilde{t}_j$ and the surface moment $\tilde{m}_j$. We suppose that the external data $f_j^{(\rho)}$, $g_j^{(\rho)}$, $\tilde{t}_j$ and $\tilde{m}_j$ are all independent of the axial coordinate.

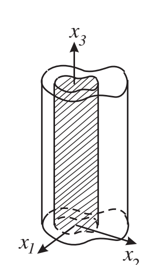

Fig. 1. A reinforced cylinder.