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The deformation of an elastic rod with a clamp sliding along a smooth and curved profile

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ABSTRACT

The design of compliant mechanisms is crucial in several technologies and relies on the availability of solutions for nonlinear structural problems. One of these solutions is given and experimentally validated in the present article for a compliant mechanism moving along a smooth curved profile. In particular, a deformable elastic rod is held by two clamps, one at each end. The first clamp is constrained to slide without friction along a curved profile, while the second clamp moves in a straight line transmitting its motion through the elastic rod to the first clamp. For this system it is shown that the clamp sliding on the profile imposes nontrivial boundary conditions (derived via a variational and an asymptotic approach), which strongly influence buckling and nonlinear structural behavior. Investigation of this behavior shows that a compliant mechanism can be designed, which gives an almost neutral response in compression. This behavior could easily be exploited to make a force limiting device. Finally a proof-of-concept device was constructed and tested showing that the analyzed mechanical system can be realized in practice and it behaves tightly to the model, so that it can now be used in the design of machines that use compliant mechanisms.

1. Introduction

Compliant mechanisms are going through a paradigm change. Where once they were part of the fixtures and fittings of mechanisms they are becoming the mechanisms themselves. This is especially true in nano- and micro- mechanics where joints, linkages and their associated bearings are difficult to make and assemble. They are also important in bioinspired systems, as biological systems are often composed of soft elements working over a large range of displacements. Advances in this field are heavily reliant on the available solutions for the non linear behavior of structural elements as well as physical proof that the theoretical models can be realized in practice. The purpose of the present article is to investigate, both theoretically and experimentally, the planar elastica with a 'non-standard constraint' applied at one of its ends, namely, a clamp that is constrained to follow a curved and frictionless profile. The influence of constraints of this type on elastic rods has been recently highlighted by Zaccaria et al. (2011), showing that a slider can introduce tensile buckling in an elastic system[1], and by Bigoni et al. (2012, 2013), demonstrating the strong influence of constraint curvature on buckling (which, by 'playing' with the sign of the curvature, may be turned from compressive to tensile) and postcritical behavior.

To illustrate the effect of a 'non-standard constraint' on structural systems, let us consider the two simple structures of length l shown in Fig. 1, both subjected to a distributed transverse load of magnitude q and simply supported at the left end (the rods are assumed inextensible and solved in the small deflection approximation). The difference between the two structures lies in the constraint applied on their right end: figure A shows a clamp

[1] Bifurcation for tensile loads was also addressed by Ziegler (1977), but a compressed element responsible of buckling is present in his example, and by Gajewski and Palej (1974), as a result of a live load. Zyczkowski (1991) concludes that under dead loading buckling is impossible when all structural elements are subject to tension. Biezeno and Grammel (1955) fail to notice that one structure considered by them as an example of multiple loadings displays tensile buckling, when subjected to a certain load. The example reported by Zaccaria et al. (2011) shows that tensile buckling of a system in which all elements are subject to tension is possible under a dead load. To the best of the authors' knowledge, and excluding a situation analyzed by Timoshenko and Gere (1936), the effect of constraint curvature on buckling load was only considered by Bigoni et al. (2012, 2013).

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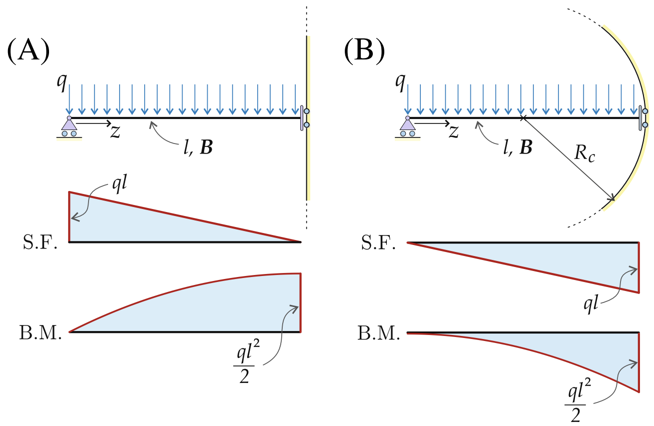

Fig. 1. Two elastic structures (the rods have a length l and are subject to a vertical, distributed load q) differing only in the curvature of the profile along which the clamp on the right end can slide (a vertical line on the left, a circle of radius $R_c = l/2$ on the right) exhibit a completely different mechanical behavior, so that, while the structure on the left is equivalent to one half of a simply-supported beam, that on the right is equivalent to a beam simply clamped on the right and free at the other end. 'S.F.' and 'B.M.' stand for 'shear force' and 'bending moment', respectively.

that is free to slide along a vertical line while figure B shows a clamp that is constrained to follow a curve, which for simplicity is circular with $R_c = l/2$. It can be appreciated from the diagrams of the bending moment (B.M.) and of the shear force (S.F.) shown in the figure that the two elastic solutions are totally different, a fact that demonstrates the strong influence of the constraint's curvature.

In order to derive the natural boundary condition that emerges from the presence of the curved constraint, it suffices to write the total potential energy of the elastic system shown in Fig. 1, expressed as a function of the transverse displacement $v(z)$,

$$\mathcal{V}(v) = \frac{1}{2} B \int_0^l \left(\frac{d^2 v(z)}{dz^2} \right)^2 dz - q \int_0^l v(z) dz, \tag{1}$$

and to take variations (subscript 'var') of the equilibrium configuration (subscript 'eq') in the form

$$v(z) = v_{eq}(z) + \epsilon v_{var}(z), \tag{2}$$

subject to the following kinematic boundary conditions at the extremities of the rod,

$$\begin{aligned} &v_{eq}(0) = 0, \quad \left. \frac{d v_{eq}(z)}{dz} \right|_{z=l} + \chi v_{eq}(l) = 0 \quad \text{and} \\ &v_{var}(0) = 0, \quad \left. \frac{d v_{var}(z)}{dz} \right|_{z=l} + \chi v_{var}(l) = 0, \end{aligned} \tag{3}$$

where χ is the signed curvature of the constraint, assumed to be constant. The first variation of the functional (1) is readily obtained as

$$\begin{aligned} \delta_\epsilon \mathcal{V}(v) = &\int_0^l \left[B \frac{d^4 v_{eq}(z)}{dz^4} - q \right] v_{var}(z) dz \\ &+ B \frac{d^2 v_{eq}(z)}{dz^2} \frac{d v_{var}(z)}{dz} \Bigg|_{z=0}^{z=l} - B \frac{d^3 v_{eq}(z)}{dz^3} v_{var}(z) \Bigg|_{z=0}^{z=l}, \end{aligned} \tag{4}$$

such that, taking into account the restrictions imposed by Eq. (3), the vanishing of $\delta_\epsilon \mathcal{V}(v)$ for any admissible displacement field $v_{var}(z)$ leads to the differential equation of the linearized elastica (not reported for brevity) and to the non-trivial boundary condition at the right end of the structure, namely,

$$\left. \frac{d^2 v(z)}{dz^2} \right|_{z=l} + \frac{1}{\chi} \left. \frac{d^3 v(z)}{dz^3} \right|_{z=l} = 0, \quad \text{or} \quad M(l) + \frac{1}{\chi} T(l) = 0, \tag{5}$$

so that the force $T(l)$, tangential to the moving clamp (and coincident now with the shear force transmitted by the elastic rod[2]), is in general not null, but related to the bending moment $M(l)$ through the curvature χ of the profile along which the clamp may slide. The fact that a reaction is present, tangential to a perfectly smooth constraint is an unexpected and noticeable effect sharing similarities with the Eshelby-like force discovered by Bigoni et al. (2014a,b, 2015), see also Bosi et al. (2014). This reaction (which is completely unexpected at first glance) passed unnoticed by Bigoni et al. (2012) (because they did not experiment a sliding clamp, but only a sliding pin), so that it is the purpose of this article to reconsider the sliding clamp condition, providing for this constraint a full theoretical and experimental validation.

The scope of the present study is: (i) to generalize the boundary condition at the curved constraint, that is Eq. (5), showing that it holds true for a profile with variable curvature and when the rod is subject to large displacements, (ii) to solve the nonlinear equations of the elastica for a rod with a clamped end movable on a circular constraint, and (iii) to experimentally show, through the realization of a proof-of-concept device, that a sliding clamp can be realized in practice to tightly follow the theory. In particular, the presented experiments refer to the case of a rectilinear elastic rod with one clamped end constrained to slide along a bi-circular 'S-shaped' profile, the same system considered by Bigoni et al. (2012), but from a purely theoretical point of view and with a no-shear assumption at the sliding profile, which is correct only for a movable pin, as shown in the present study. The influence of the constraint's curvature is shown both on the critical loads and on the post-critical behavior, obtained by direct integration of the nonlinear equation of the elastica. An interesting finding is that the postcritical response in compression is 'almost neutral', in the sense that the load changes very little with increasing displacement of the clamped end, a feature that could be exploited in the realization of a force limiter device, which could for instance be employed in the design of shock absorbers or security belts.

2. The elastica with a sliding clamp at one end: derivation of boundary conditions

The boundary condition (5), relating shear force, bending moment and curvature at the sliding constraint can be obtained both with a variational approach and with an asymptotic approach. In fact, the clamp can be replaced by two rollers, close to each other and joined by a rigid bar, in the limit when the length of the bar (and therefore the distance between the two rollers) tends to zero. The boundary condition naturally emerges in the former approach, while the latter represents the key for applications, since it will be experimentally proven that a moving clamp can be realized with two rollers rigidly joined at a small distance from each other.

2.1. Variational approach

Consider an elastic inextensible rod of length l and bending stiffness B that is clamped at its left end, while the other end is constrained by another clamp which is free to slide along a frictionless and curved profile parametrically described through ξ as $\mathbf{x}^c(\xi)$, see Fig. 2 for a sketch of the structure.

[2] This coincidence is not verified if the clamp is connected to the end of the elastic rod through an elastic hinge.