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# Singular perturbations and cloaking illusions for elastic waves in membranes and Kirchhoff plates

I.S. Jones [a,*], M. Brun [b], N.V. Movchan [c], A.B. Movchan [c]

[a] School of Engineering, John Moores University, Liverpool L3 3AF, UK

[b] Dipartimento di Ingegneria Meccanica, Chimica e dei Materiali, Universitá di Cagliari, Piazza d'Armi, I-09123 Cagliari, Italy

[c] Department of Mathematical Sciences, University of Liverpool, Liverpool L69 3BX, UK

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ABSTRACT

A perturbation approach is used for analysis of a near-cloak in shielding a finite scatterer from an incident flexural wave. The effect of the boundary conditions on the interior surface of the cloaking layer is analysed in detail, based on the explicit analytical solutions of a wave propagation problem for a membrane as well as a Kirchhoff flexural plate. It is shown that the Dirichlet boundary condition on the interior contour of the cloak significantly reduces the cloaking action in the membrane case, and it also makes cloaking impossible for flexural waves in a Kirchhoff plate.



## 1. Introduction

The cloaking of acoustic and electromagnetic waves has been extensively developed in terms of theoretical design and practical implementation in papers (Pendry et al., 2006; Schurig et al., 2006; Leonhardt, 2006; Cummer and Schurig, 2007; Norris, 2008; Chen and Chan, 2010; Guenneau et al., 2011; Kadic et al., 2013; Zigoneanu et al., 2014). These publications are based on the transformation optics approach, that may employ a non-conformal push-out map. The theory equally applies to linear waterwave problems, as discussed in Farhat et al. (2008).

Elastic cloaking, both for vector problems of elasticity and for flexural waves in Kirchhoff plates, brings new challenges regarding the physical interpretation of equations and boundary conditions in the cloaking region. Indeed, this is now well understood in the context of a lack of invariance of the governing equations of elasticity with respect to a non-conformal cloaking transform. Analysis of elastic cloaking problems had been reported in Milton et al. (2006), Brun et al. (2009), Norris and Shuvalov (2011), and Norris and Parnell (2012).

In the recent papers (Farhat et al., 2009a,b; Stenger et al., 2012; Brun et al., 2014; Colquitt et al., 2014), an advance has been made in the theoretical analysis, design and the physical interpretation of a cloak for flexural waves in Kirchhoff plates. Although it is known that membrane waves (solutions of the Helmholtz equation) drive the wave propagation in Kirchhoff plates (see, for example, McPhedran et al. (2009), Antonakakis and Craster (2012), Brun et al. (2012), Carta et al. (2014), and McPhedran et al. (2014)), the interface and boundary conditions provide coupling between solutions of the Helmholtz and modified Helmholtz equations in problems of scattering of flexural waves. In Brun et al. (2014) and Colquitt et al. (2014), we have analysed in detail the transformed plate equation, and have shown that in the cloaking region the physical interpretation requires the presence of prestress and body force terms. Subject to such an allowance, the cloaking of defects in Kirchhoff plates becomes feasible and well understood. In the present paper, we make an emphasis on the important issue of the choice of boundary conditions, which are set at the interior contour of the cloak.

In cloaking transformation problems, it is common that boundary conditions on the interior boundary of a cloaking region are not addressed (Farhat et al., 2009a,b). This matter is rarely discussed and numerical simulations for cloaking are commonly presented without additional comments regarding these boundary conditions. The reason is that for a singular map, which "stretches" a hole of zero radius into a finite disk, in the unperturbed configuration there is no boundary. Nevertheless, in the subsequent numerical computations the singularity of the material constants in the cloaking region is replaced by regularised finite values. Namely, the boundary conditions are required for the numerical computations and usually natural boundary conditions that follow from

* Corresponding author.
*E-mail address:* i.s.jones@ljmu.ac.uk (I.S. Jones).

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the variational formulation are chosen, i.e. these are Neumann boundary conditions on the interior contour of the cloaking region (Norris and Parnell, 2012; Parnell et al., 2012).

One can ask a naive question related to an illusion rather than cloaking. For example, if one has a carrot cake, would it be possible to make it look like a fairy cake instead. In turn, could one make a large void in a solid look smaller? Purely naively, the latter can be addressed through a geometrical transformation in polar coordinates:

$$r = \alpha_1 + \alpha_2 R, \quad \theta = \Theta,$$

where

$$\alpha_1 = \frac{R_2(R_1 - a)}{R_2 - a}, \quad \alpha_2 = \frac{R_2 - R_1}{R_2 - a}.$$

and

$$a \leqslant R \leqslant R_2, \quad R_1 \leqslant r \leqslant R_2.$$

with $a$ being a small positive number and $(R, \Theta)$ being the coordinates before the transformation and $(r, \theta)$ after the transformation, and $R_1,\ R_2$ being positive constants, such that $R_1 < R_2$. Such a transformation, within the ring $R_1 < r < R_2$, would correspond to a radial non-uniform stretch, and assuming that outside the disk $r > R_2$ there is no deformation (i.e. $r = R$), we obtain a coating that would lead to an illusion regarding the size of a defect in a solid. Push-out transformations of such a type have been widely used, and have been applied to problems of cloaking, for example, in papers (Leonhardt, 2006; Greenleaf et al., 2003).

Formally, if $u$ represents an undistorted field in the exterior of a small void, of radius $a$ and we assume the series representation

$$u(R, \Theta) = \sum_{n=-\infty}^{\infty} V_n(R) W_n(\Theta),$$

with $V_n$ and $W_n$ being basis functions, then the distortion, represented by the fields $u_1$ and $u_2$ can be described through a "shifted" series representation, as follows

$$u = u_1(r, \theta) = \sum_{n=-\infty}^{\infty} V_n\left(\frac{r - \alpha_1}{\alpha_2}\right) W_n(\theta), \quad \text{when } R_1 < r < R_2$$

and

$$u = u_2(R, \Theta) = \sum_{n=-\infty}^{\infty} V_n(R) W_n(\Theta), \quad \text{when } r > R_2.$$

Such a transformation delivers an illusion, which makes a finite circular void of radius $R_1$, look like a small void of radius $a$.

The above argument may appear to be simplistic, but we are going to show that it works exactly as described, for a class of problems governed by the Helmholtz operator and by the equations of vibrating Kirchhoff plates. In this cases, the basis functions $V_n$ and $W_n$ are chosen accordingly, and are written in the closed form in the main text of the paper.

We consider the problem in the framework of singular perturbations and, instead of regularising the singular values of material parameters after the cloaking transformation, we begin by introducing a small hole and apply the cloaking transformation to a region containing such a small hole with appropriate boundary conditions already chosen. This is consistent with the analysis presented in Kohn et al. (2008) and Colquitt et al. (2013).

In the present paper, we show that the cloaking problem is closely related to that for an infinite body containing a small hole of radius $a$ where we set a boundary condition of either the Dirichlet (clamped boundaries) or Neumann (free-edge boundaries) type. This is done for both membrane waves governed by the Helmholtz equation and flexural waves that occur in Kirchhoff plates. Analytical solutions are presented for the cloaking problems and it is shown that the degree of cloaking is highly dependent on the boundary condition on the interior contour of the cloaking region. In a plate, in the presence of an incident plane wave along the $x$-axis, the scattered flexural field, $u_s$, outside the cloaking region, when $a \to 0$ with a clamped interior boundary, has the asymptotic representation at a sufficiently large distance $R$ from the centre of the scatterer

$$u_s \sim -\sqrt{\frac{2}{\pi\beta R}} \exp\left(i\left(\beta R - \frac{\pi}{4}\right)\right), \quad \text{as } \beta R \to \infty, \tag{1}$$

where $\beta^4 = \rho h \omega^2 / D_0$ with radian frequency $\omega$, plate flexural rigidity $D_0$, plate thickness $h$ and density $\rho$. This immediately suggests the absence of any cloaking action as the above asymptotic representation corresponds to a finite point force initiated by a rigid pin at the origin (see Evans and Porter, 2007; McPhedran et al., 2009). On the contrary, the free-edge interior boundary in the cloaking layer for a flexural plate produces high-quality cloaking action, and the corresponding asymptotic representation of the scattered field $u_s$ at sufficiently large $\beta R$ becomes

$$u_s \sim \sqrt{\frac{2}{\pi\beta R}} \exp\left(i\left(\beta R - \frac{\pi}{4}\right)\right) \frac{\pi i \nu}{4(1-\nu)} (\beta a)^2 = O(\beta a)^2,$$
$$\text{as} \quad \beta a \to 0 \quad \text{and} \quad \beta R \to \infty. \tag{2}$$

This confirms that the regularisation algorithm that refers to a small free-edge hole produces a scattered field proportional to the area of this small hole, and it tends to zero as $\beta a \to 0$.

For the "cloaking type" map considered in this paper, the original hole does not have to be small, and we will refer to a "cloaking illusion", which results in an object being mimicked by an obstacle of a different size.

The structure of the paper is as follows. In Section 2 we introduce the notion of a cloaking transformation. Section 3 addresses the singular perturbation approach for an elastic membrane. In particular, in Section 3.1 we include an analytical solution, together with asymptotic estimates, for the model problem of scattering of membrane waves from a small circular scatterer. Section 3.2 shows the relationship between the model problem for a body with a small scatterer and the full cloaked problem. Section 4 presents the analytical solution and the asymptotic analysis of scattering of flexural waves in a Kirchhoff plate for the biharmonic cloaking problem. The concluding remarks are included in Section 5, where we summarise the fundings regarding the cloaking action for different types of boundary conditions at the interior boundary of the cloaking region.

## 2. Cloaking transformation

We aim to consider flexural waves in thin elastic plates. As shown in Brun et al. (2014) and Colquitt et al. (2014), after the cloaking transformation such waves can be interpreted as time-harmonic flexural displacements in a pre-stressed anisotropic thin plate. Firstly, we review the cloaking transformation procedure and then we discuss the transformed equations and their physical interpretation.

### 2.1. Push-out transformation

A non-conformal transformation is introduced to define a cloaking coating. We use the notations $\boldsymbol{X} = (R, \Theta)^T$ and $\boldsymbol{x} = (r, \theta)^T$ for coordinates before the transformation and after the transformation, respectively. Here $(R, \Theta)$ are polar coordinates. With reference to Pendry et al. (2006), Schurig et al. (2006), Leonhardt (2006), Greenleaf et al. (2003), and Norris (2008), we use the radial invertible "push-out" map $\mathcal{F} : \boldsymbol{X} \to \boldsymbol{x}$ defining the new stretched