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Revisiting the identification of generalized Maxwell models from experimental results

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ABSTRACT

Linear viscoelastic material behavior is often modeled using a generalized Maxwell model. The material parameters, i.e. relaxation times and elastic moduli, of the Maxwell elements are determined from either a *relaxation* or a *Dynamical Mechanical Analysis* (DMA) experiments. The underlying mathematical problem is known to be ill-posed, which means that uniqueness of the identification is not assured and that small errors in the initial data will conduct to high discrepancies in the identified parameters. The standard technique to remove the ill-posedness is to chose a priori a series of relaxation times and to identify only the moduli. The aim of this paper is to propose two techniques to identify an optimal series of relaxation times. In the case of the relaxation experiment relaxation times will be optimized from the numerical integration of the measured relaxation spectrum. In the case of the DMA experiments we show that mathematical results obtained by Krein and Nudelmann can be used to determine the complete series of relaxation times. The methods are illustrated by identification examples using both artificial and experimental data. The results show that the methods provide a good match of the identified models in term of relaxation or complex moduli.

1. Introduction

A large number of materials including rubber, polymers, composites, concrete, etc. have a viscoelastic mechanical behavior which is often represented using a generalized Maxwell model (Findley et al., 1976). The model presents several merits: it is simple, robust and can be identified from *relaxation* or *Dynamical Mechanical Analysis* (DMA) experiments. Moreover it can cover a large range of characteristic times in both experiments. Application of the generalized Maxwell model cover different classes of polymers amorphous or cross-linked polymers (Brinson, 2008), polydisperse, high density polyethylene (Otegui et al., 2013) etc. Other applications cover concrete materials as in Park and Kim (2001).

Further extension based on the Linear generalized Maxwell model are cover nonlinear viscoelastic material behaviors, where different parameters like the elastic moduli or the relaxation times will further depend on different parameters. Let us cite, the curing dependent relaxation moduli proposed in Zarrelli et al. for epoxy materials or a prestrain dependent complex modulus proposed for propellant in Thorin et al. (2013,).

The identification of the relaxation spectrum of a viscoelastic system, corresponds to the determination of the relaxation kernel in an integral equation and is denoted as a Fredholm integral equation of the first kind. The problem has attracted a lot of attentions during the last decades due to its inherent difficulties. It is mathematically ill-posed, implying that the identification of the kernel is not uniquely assured and that small errors in the initial data will conduct to high discrepancies in the identified kernel. Within the recent mathematical literature, we can cite the work of Grasselli (1994), Janno and Von Wolfersdorf (1997), Von Wolfersdorf (1993), Cavaterra and Grasselli (1997), which recovered the relaxation spectrum by reducing the problem to a nonlinear Volterra integral equation using a Fourier method to solve the direct problem and by applying the contraction principle. Further results reveled that the problem can also be solved in a heterogeneous medium, as in Lorenzi (1999), Lorenzi and Romanov (2006) or recently de Buhanand and Osses (2010) meaning that a spatial material heterogeneity can also be recovered if specific conditions are satisfied.

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In order to eliminate the ill-posedness of the identification problem, a standard technique consists of defining a priori the characteristic times for the Prony's series and pursuing the identification only for the elastic moduli of the different elements. This technique is described for example in Honerkamp (1989), Honerkamp and Weese (1990) for the DMA experiments, where the identification is performed on the complex moduli of the material. A recent application example is given in Diani et al. (2012), where a generalized Maxwell model is identified to describe the viscoelastic behavior of shape memory polymers. For the relaxation experiment a dual method is proposed in Baumgaertel and Winter (1992), Gerlach and Matzenmiller (2005) where the relaxation modulus is identified. Other techniques are based on different numerical schemes to tackle this problem, as for example the combination of Laplace transform and Padé approximants reported in Carrot and Verney (1996), or the cumulative relaxation spectrum proposed in Xiao et al. (2013).

The aim of this paper is to improve the existing techniques by proposing a novel way to determine a series of characteristic times. In the case of a DMA experiment, the improvement is based on a mathematical result given by Krein and Nudelman (1998), which permits to identify the relaxation times as the zeros of two complex functions constructed from the measured data. This problem setting does not eliminate the ill-posedness of the initial problem. Two algorithmic matrices have to be positive definite in order to numerically solve the problem and this is realized through a regularization technique. However this imposes more natural restrictions in the problem when compared with an artificial set of characteristic times. In the case of the relaxation experiments, the idea is to optimize the a priori set of relaxation time by imposing a closer representation of the relaxation function by reanalyzing the Riemann integration process. Results for both experiments provide a smaller number of branches in the generalized Maxwell model than traditional methods and keep the quality of the representation.

The paper starts with an overview of the viscoelastic generalized Maxwell model where notations and general concepts are introduced, as the continuous spectrum or the discrete Prony's series. The third section presents the standard identification technique of parameters from DMA experiments as presented by Honerkamp (1989), Honerkamp and Weese (1990) and the theoretical results of Krein and Nudelman (1998) as well as the proposed identification algorithm. The discussion continues with the identification method of parameters from relaxation test as proposed in Baumgaertel and Winter (1992), Gerlach and Matzenmiller (2005) and the proposed optimal identification of relaxation times. The two methods are illustrated in the last chapter by a series of examples: first using artificial data, which also permits to investigate the influence of the noise and second using experimental data from literature and measurements.

2. Viscoelasticity and Prony's series

Let us start by recalling some concepts in linear viscoelasticity in order to define the notations and the basic equations used in this study.

The viscoelastic constitutive behavior can be represented in the time domain, according to Markovitz and Hershel (1977), by relating histories of stresses, σ, and strains, ε, through the integral equation:

$$\sigma(t) = \int_{-\infty}^{t} E(t-\tau)\frac{d\varepsilon(\tau)}{d\tau}d\tau \tag{1}$$

where E denotes the relaxation kernel.

The dual representation of constitutive equation in the frequency domain, relates the Fourier transform of stress and strains, denoted as σ^* and ε^* respectively, by a linear equation:

$$\sigma^*(\omega) = E^*(\omega)\varepsilon^*(\omega) \tag{2}$$

$E^*(\omega)$ is the complex modulus depending of the frequency ω and obtain through the same Fourier transform as stresses or strains.

The two equations in the time domain or frequency domain, are equivalent and can be obtained the a direct or inverse Fourier Transform denoted with a $*$.

The main constitutive unknown is the relaxation spectrum $H(\tau)$ (Findley et al., 1976) which is related to the relaxation modulus $E(t)$ by:

$$E(t) = E_0 + \int_{-\infty}^{\infty} H(\tau)e^{-\frac{t}{\tau}}d\ \ln(\tau) \tag{3}$$

and to the dynamical modulus $E^*(\omega) = E'(\omega) + iE''(\omega)$ by:

$$E'(\omega) = E_0 + \int_{-\infty}^{\infty} H(\tau)\frac{\omega^2\tau^2}{1+\omega^2\tau^2}d\ \ln(\tau) \quad E^{Prime}(\omega)$$
$$= \int_{-\infty}^{\infty} H(\tau)\frac{\omega\tau}{1+\omega^2\tau^2}d\ \ln(\tau) \tag{4}$$

For practical reasons it is convenient to use a model, where the continuous spectrum of relaxation $H(\tau)$ is replaced with a finite spectrum $\hat{H}(\tau)$ (Eq. (5)). This later is interpreted as simple rheological elements, springs and dampers, and is denoted generalized Maxwell model (see Fig. 1). It mathematical description is the finite Prony's series (τ_i, E_i) and the spectrum becomes:

$$\hat{H}(\tau) = \sum_{i=1}^{n} E_i\ \delta\left(1 - \frac{\tau}{\tau_i}\right) \tag{5}$$

where δ is the Dirac function. The relaxation time τ_i associated to the element i is related to the characteristic time of the spring-damper element, and is defined as the ratio of the viscosity over the elastic moduli, i.e. $\tau_i = \frac{\eta_i}{E_i}$.

This representation is often used in finite element models, see Simo and Hughes (2008) for the time integration within a finite element code.

In the discrete case of Prony series, the relaxation modulus $E(t)$ is represented as:

$$E(t) = E_0 + \sum_{i=1}^{n} E_i\ e^{-\frac{t}{\tau_i}} \tag{6}$$

where E_0 represents the stiffness of the model at large times and n denotes the number of branches of the generalized Maxwell model. In the frequency domain, the dynamical modulus $E^*(\omega)$ becomes:

$$E'(\omega) = E_0 + \sum_{i=1}^{n}\frac{E_i\ \omega^2\tau_i^2}{1+\omega^2\tau_i^2} \quad E''(\omega) = \sum_{i=1}^{n}\frac{E_i\ \omega\tau_i}{1+\omega^2\tau_i^2} \tag{7}$$

Let us now consider that mechanical experiments such as relaxation test or cyclic loading test using a Dynamical Mechanical Analyzer (DMA) provide a data series representing a continuous

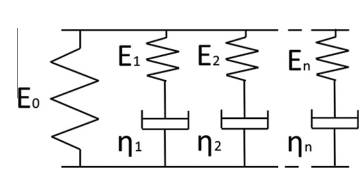

Fig. 1. A schematic representation of the generalized Maxwell model as a parallel association of n Maxwell units, i.e. linear spring and damper (E_i, τ_i) and a linear spring of stiffness E_0.