ARTICLE IN PRESS

International Journal of Solids and Structures xxx (2014) xxx–xxx

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

On non-singular crack fields in Helmholtz type enriched elasticity theories

Markus Lazar [a,*], Demosthenes Polyzos [b]

[a] Heisenberg Research Group, Department of Physics, Darmstadt University of Technology, Hochschulstr. 6, D-64289 Darmstadt, Germany
[b] Department of Mechanical and Aeronautical Engineering, University of Patras, GR-26500 Patras, Greece

ARTICLE INFO

Article history:
Received 17 April 2013
Received in revised form 12 December 2013
Available online xxxx

Keywords:
Cracks
Dislocations
Fracture mechanics
Nonlocal elasticity
Gradient elasticity

ABSTRACT

Recently, simple non-singular stress fields of cracks of mode I and mode III have been published by Aifantis (2009, 2011), Isaksson and Hägglund (2013) and Isaksson et al. (2012). In this work we investigate the physical meaning and interpretation of those solutions and if they satisfy important physical conditions (equilibrium, boundary and compatibility conditions).

1. Introduction

During the last years, some non-singular crack fields have been published in the literature (Aifantis, 2009, 2011; Isaksson and Hägglund, 2013; Isaksson et al., 2012) neglecting equilibrium, boundary and compatibility conditions. The aim of that research was the regularization of the classical singular crack fields. In fact, the non-singular crack fields are zero at the crack tip. However, not any equilibrium condition was used by Aifantis (2011), Isaksson and Hägglund (2013) and Isaksson et al. (2012). Therefore, it is doubtful if their results are correct from the point of view of fracture mechanics.

On the other hand, Ari and Eringen (1983), Eringen and Suresh (1983) and Eringen (1984) (see also Eringen, 2002) investigated cracks in the framework of nonlocal elasticity of Helmholtz type in the 80s. Eringen (1984, 2002) found a non-singular stress of a mode III crack zero at the crack tip. For the mode I crack problem, using appropriate boundary conditions, Ari and Eringen (1983) (see also Eringen, 2002) found a non-singular stress finite at the crack tip and becoming zero inside the crack. A regularization procedure was also discussed for crack curving in Eringen and Suresh (1983).

The aim of this paper is to show that the recent crack solutions given by Aifantis (2009, 2011), Isaksson et al. (2012) and Isaksson and Hägglund (2013) cannot be the correct solutions of a nonlocal or gradient compatible elastic fracture mechanics problem. Therefore, the modest goal of the present paper is not to give new solutions but rather to discuss existing and recently given crack solutions using gradient enhanced elasticity theories.

The paper is organized as follows: Section 2 provides the basics of the theories of nonlocal elasticity and strain gradient elasticity. Next, Section 3 explains why the non-singular mode III crack solution given by Aifantis (2009, 2011) cannot be considered as a solution of a nonlocal and gradient elastic fracture mechanics problem. The same is explained in Section 4 for the mode I crack solution given by Aifantis (2011), Isaksson et al. (2012) and Isaksson and Hägglund (2013). Finally, in Section 5 a possible way-out for the physical interpretation of the non-singular solutions is discussed.

2. Theoretical framework

In this section we outline the basics of the theories of nonlocal elasticity and gradient elasticity.

2.1. Theory of nonlocal elasticity of Helmholtz type

In the theory of nonlocal elasticity (e.g., Eringen, 2002, 1983), the so-called nonlocal stress tensor t_{ij} is defined at any point $\boldsymbol{x}$ of the analyzed domain of volume V as

$$t_{ij}(\boldsymbol{x}) = \int_V \alpha(|\boldsymbol{x}-\boldsymbol{y}|)\sigma^0_{ij}(\boldsymbol{y})\,\mathrm{d}V(\boldsymbol{y}), \tag{1}$$

where $\alpha(|\boldsymbol{x}-\boldsymbol{y}|)$ is a nonlocal kernel and σ^0_{ij} is the stress tensor of classical elasticity defined at the point $\boldsymbol{y} \in V$ as

$$\sigma^0_{ij}(\boldsymbol{y}) = \lambda\,\delta_{ij}e^0_{kk}(\boldsymbol{y}) + 2\mu e^0_{ij}(\boldsymbol{y}), \tag{2}$$

* Corresponding author. Tel.: +49 0 6151163686.
E-mail addresses: lazar@fkp.tu-darmstadt.de (M. Lazar), polyzos@mech.upatras.gr (D. Polyzos).

0020-7683/$ - see front matter © 2014 Elsevier Ltd. All rights reserved.
http://dx.doi.org/10.1016/j.ijsolstr.2014.01.002

Please cite this article in press as: Lazar, M., Polyzos, D. On non-singular crack fields in Helmholtz type enriched elasticity theories. Int. J. Solids Struct. (2014), http://dx.doi.org/10.1016/j.ijsolstr.2014.01.002

with λ, μ are the Lamé constants, δ_{ij} is the Kronecker delta and e^0_{ij} denotes the classical strain tensor, which is the symmetric part of the classical distortion tensor

$$e^0_{ij} = \frac{1}{2}\left(\beta^0_{ij} + \beta^0_{ji}\right). \tag{3}$$

We employ a comma to indicate partial derivative with respect to rectangular coordinates x_j, i.e. $t_{ij,j} = \frac{\partial t_{ij}}{\partial x_j}$. As usual, repeated indices indicate summation.

In absence of body forces, the nonlocal stress tensor satisfies the equilibrium condition

$$t_{ij,j} = 0, \tag{4}$$

which means that the stress is self-equilibrated. In addition, the classical stress tensor fulfills the equilibrium equation of classical elasticity

$$\sigma^0_{ij,j} = 0. \tag{5}$$

If the nonlocal kernel function $\alpha(|\boldsymbol{x} - \boldsymbol{y}|)$ is the Green function (fundamental solution) of the differential operator $L = 1 - \ell^2\Delta$, i.e.

$$(1 - \ell^2\Delta)\alpha(|\boldsymbol{x} - \boldsymbol{y}|) = \delta(\boldsymbol{x} - \boldsymbol{y}), \tag{6}$$

with ℓ, Δ, δ being a characteristic length scale ($\ell \geqslant 0$), the Laplacian and the Dirac delta function, respectively, then the integral relation (1) reduces to the inhomogeneous Helmholtz equation

$$(1 - \ell^2\Delta)t_{ij} = \sigma^0_{ij}, \tag{7}$$

where the classical stress is the source for the nonlocal stress. The natural boundary condition reads

$$t_{ij}n_j = \hat{t}_i, \tag{8}$$

where n_i and $\hat{t}_i$ represent the normal to the external boundary and the prescribed boundary tractions, respectively. In nonlocal elasticity, no nonlocal strain exists. Thus, using a nonlocal kernel, being a Green function, yields a differential equation for t_{ij} instead of an integrodifferential equation in the 'strongly' nonlocal theory with seemingly and physically equivalent solution at the output. In such a 'weakly' nonlocal elasticity the concept of a gradient theory might be used (Maugin, 1979, 2012). The 'weakly' nonlocal theory of elasticity represented by Eqs. (4)–(7) is called of Helmholtz type because the Helmholtz operator, $L = 1 - \ell^2\Delta$, enters in the form of Eqs. (6) and (7).

It was pointed out by Eringen and Suresh (1983) that the stress field t_{ij} of a crack is obtained by solving Eq. (7), subject to regularity conditions, i.e., t_{ij} must be bounded at the crack tip and at infinity. This is borne out from the problems of non-singular dislocations (Eringen, 2002). At large distance from the crack tip, the classical solution will approximate the stress field well, namely if $\ell \to 0$, Eq. (7) gives $t_{ij} \to \sigma^0_{ij}$. This also suggests that one may obtain a full solution of Eq. (7) and match it to the outer solution σ^0_{ij} in order to obtain a non-singular solution.

2.2. Theory of gradient elasticity of Helmholtz type

In the theory of gradient elasticity (see, e.g., Mindlin, 1964; Mindlin and Eshel, 1968; Eshel and Rosenfeld, 1970; Jaunzemis, 1967), the equilibrium condition is given by

$$\tau_{ij,j} - \tau_{ijk,jk} = 0, \tag{9}$$

where τ_{ij} is the Cauchy-like stress tensor and τ_{ijk} is the so-called double-stress tensor. It can be seen in Eq. (9) that the Cauchy-like stress tensor τ_{ij} is, in general, not self-equilibrated. The natural boundary conditions in strain gradient elasticity are much more complicated than the corresponding ones in nonlocal elasticity; they read (see, e.g., Mindlin and Eshel, 1968; Jaunzemis, 1967)

$$\left.\begin{array}{l}(\tau_{ij} - \partial_k\tau_{ijk})n_j - \partial_j(\tau_{ijk}n_k) + n_j\partial_l(\tau_{ijk}n_k n_l) = \bar{t}_i \\ \tau_{ijk}n_j n_k = \bar{q}_i\end{array}\right\} \quad \text{on} \quad \partial\Omega, \tag{10}$$

where $\bar{t}_i$ and $\bar{q}_i$ are the prescribed Cauchy traction vector and the prescribed double stress traction vector, respectively. Moreover, $\partial\Omega$ is the smooth boundary surface of the domain Ω occupied by the body.

In a simplified version of strain gradient elasticity, called gradient elasticity of Helmholtz type (e.g., Lazar and Maugin, 2005; Lazar, 2013; Polyzos et al., 2003), the double stress tensor is nothing but the gradient of the Cauchy-like stress tensor multiplied by ℓ^2

$$\tau_{ijk} = \ell^2\tau_{ij,k} \tag{11}$$

and the Cauchy-like stress tensor reads

$$\tau_{ij} = C_{ijlk}\beta_{kl}, \tag{12}$$

where β_{ij} denotes the elastic distortion tensor and C_{ijkl} is the tensor of the elastic moduli given by

$$C_{ijkl} = \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \lambda\delta_{ij}\delta_{kl}. \tag{13}$$

Substituting Eq. (11) into (9), Eq. (9) simplifies to the following partial differential equation (pde) of 3rd order

$$(1 - \ell^2\Delta)\tau_{ij,j} = 0 \tag{14}$$

and, using Eq. (12), Eq. (14) reads in terms of the elastic distortion tensor

$$(1 - \ell^2\Delta)C_{ijkl}\beta_{kl,j} = 0. \tag{15}$$

Following Jaunzemis (1967), the polarization of the Cauchy-like stress, sometimes called 'total stress tensor', is defined by

$$\sigma_{ij} := (1 - \ell^2\Delta)\tau_{ij}. \tag{16}$$

Then the equilibrium condition (14) reads in terms of the total stress tensor

$$\sigma_{ij,j} = 0. \tag{17}$$

On the other hand, using the so-called 'Ru–Aifantis theorem' (Ru and Aifantis, 1993) in terms of stresses, Eq. (14) can be written as an equivalent system of pdes of 1st order and of 2nd order, namely

$$\sigma^0_{ij,j} = 0, \tag{18}$$

$$(1 - \ell^2\Delta)\tau_{ij} = \sigma^0_{ij}, \tag{19}$$

where σ^0_{ij} is the classical stress tensor. Eqs. (18) and (19) also play the role of the basic equations in Aifantis' version of gradient elasticity (see, e.g., Askes and Aifantis, 2011). Using the 'Ru–Aifantis theorem', the total stress tensor σ_{ij} is identified with the classical stress tensor σ^0_{ij}:

$$\sigma_{ij} \equiv \sigma^0_{ij}. \tag{20}$$

The so-called 'Ru–Aifantis theorem' is a special case of a more general technique well-known in the theory of partial differential equations (see, e.g., Vekua, 1967). Moreover, the 'Ru–Aifantis theorem' is restricted only to situations involving a body of infinite extent (with no need to enforce boundary conditions). In the presence of boundary conditions, the 'Ru–Aifantis theorem' is no longer valid and can lead to erroneous solutions. Therefore, it is questionable if the 'Ru–Aifantis theorem' should be used in the construction of crack solutions in gradient elasticity. In physics, such a method of the reduction of the order of higher order field equations is known and used in the so-called Bopp–Podolsky theory (Bopp, 1940; Podolsky, 1942), which is the gradient theory