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Jump conditions for strings and sheets from an action principle

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ABSTRACT

I present conditions for compatibility of velocities, conservation of mass, and balance of momentum and energy across moving discontinuities in inextensible strings and sheets of uniform mass density. The balances are derived from an action with a time-dependent, non-material boundary, and reduce to matching of material boundary conditions if the discontinuity is stationary with respect to the body. I first consider a point discontinuity in a string and a line discontinuity in a sheet, in the context of classical inertial motion in three Euclidean dimensions. I briefly comment on line discontinuities terminating in point discontinuities in a sheet, discontinuous line discontinuities in a sheet, and an approach to dynamic fracture that treats a crack tip in a sheet as a time-dependent boundary point. I provide two examples of general solutions for conservative sheet motions near a line discontinuity. The approach also enables treatment of actions depending on higher derivatives of position; I thus derive balances for an *elastica* which are applicable to moving contact problems.

1. Introduction

The dynamics of strings and sheets offer many surprises (Hanna and King, 2011; Cambou et al., 2012; Coteron; Judd; Mould, 2013; Biggins, 2014; Virga, 2014; Rennie, 1972; Calkin, 1989; Schagerl et al., 1997; Tomaszewski et al., 2006; Hamm and Géminard, 2010; Taneda, 1968; Bejan, 1982; Guven et al., 2013). Their motion unregularized by any resistance to bending, these perfectly flexible, yet inextensible bodies may develop persistent kinks and other discontinuities in their shapes. Embedded in three-dimensional space, they come into partial contact with steric, frictional, and adhesive obstacles, and thereby experience discontinuous external applied forces. Thus, the mechanics of such discontinuities should be studied.

In this paper, I consider moving discontinuities in one- and two-dimensional flexible bodies (see Figs. 1 and 2). The bodies are modeled as inextensible curves and surfaces. Physical examples of such discontinuities include peeling fronts of adhesive tapes and coatings (Ericksen, 1998; Burridge and Keller, 1978; Cortet et al., 2013; Hure and Audoly, 2013), lift-off points of chains and ropes moving around pulleys or table edges (Rennie, 1972; Prato et al., 1982; Calkin, 1989; Cambou et al., 2012), pick-up points of chains from piles or rigid surfaces (Cayley, 1857; Hanna and King, 2011; Biggins, 2014; Virga, 2014; Virga, 2015), propagating impacts in cables and membranes (Ringleb, 1957; Cristescu, 1964; Beatty and Haddow, 1985; Yokota et al., 2001; Tomaszewski et al., 2006; Hanna and Santangelo, 2012; Vandenberghe and Villermaux, 2013; Kanninen and Florence, 1967; Farrar, 1984; Haddow et al., 1992; Albrecht and Ravi-Chandar, 2014), geometrically complex propagating kinks in a windblown flag or the tubular body and arms of an Airdancer® (airdancers.com/about/), brittle cracks, tears, and cuts in sheet materials (Burridge and Keller, 1978; Roman, 2013; Vandenberghe and Villermaux, 2013), and groove structures in impressed bladders. It also seems likely that kinks may form in the transverse waves resulting from hairpin turn maneuvers of towed cables (Ivers and Mudie, 1973; Sanders, 1982; Matuk, 1983). In some of these examples, the discontinuity is an idealization that allows one to ignore a regularizing length scale, such as the body thickness, and with it higher-order terms in equations of motion, such as bending forces. When considering an elastic sheet, it may be easier to treat membrane equations and a jump condition than to explicitly examine an internal bending boundary layer and perform asymptotic matching. Thus, jump conditions may be an effective tool in studies of thin elastic bodies. For dynamic partial contact problems involving continua, the correct jump conditions are essential; one cannot simply apply static boundary conditions.

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Fig. 1. Real systems with discontinuous, or approximately discontinuous, moving features. Left: A kink moving upward in a tubular Airdancer® (airdancers.com/about/) membrane (still from a film courtesy R.B. Warner). Center: A propagating impact in an aircraft arresting cable (commons.wikimedia.org/wiki/File:AircraftCarrier3-wire.jpg). Right: A chain falling onto a table (still from a film courtesy D. Aliaj and R.B. Warner). The table, or the pile of chain on the table, acts as a positive supply of stress (momentum) and a negative supply of power (energy).

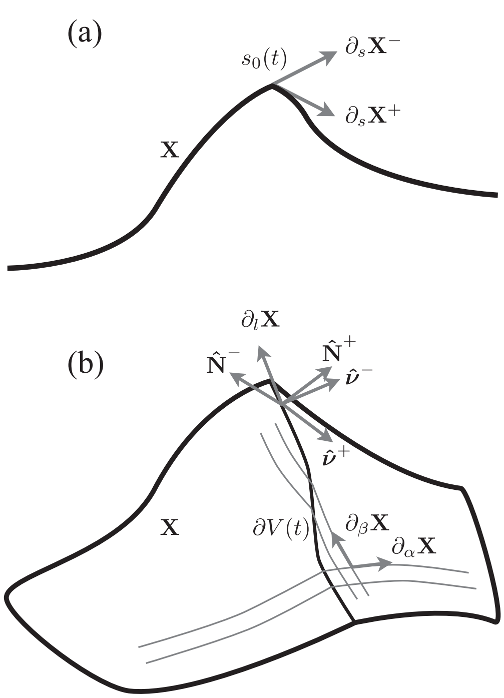

Fig. 2. Bodies **X** with discontinuities, here drawn as first order in derivatives of position. (a) Tangent vectors at a point discontinuity in a string. (b) Tangent vectors and Darboux frames at a line discontinuity in a sheet, and coordinate lines and tangents in the bulk. Details and definitions in text.

The first conditions that must hold across a discontinuity are compatibility of velocities and some body derivatives of position. A treatment of surfaces generates the equivalents of Hadamard and Darmois compatibility, which differ from the classic cases because both the discontinuity and its embedding surface are of nontrivial codimension with respect to another embedding space, $\mathbb{E}^3$.

Next I consider mass conservation, a topic which is made considerably easier by the presumption that the bodies are inextensible and isometric to a sufficiently smooth configuration. This justifies the use of global material coordinates on the body which facilitate calculations.

Then I turn to balances of momentum and energy. The balances are derived from an action with a time-dependent, non-material boundary, rather than from a set of conservation laws in weak form as would be traditional in continuum mechanics. For this reason, the formulation shares some conceptual ground with variable-mass problems (McIver, 1973), such as axially moving belts between supports (Wickert and Mote, 1988; Lee and Mote, 1997), yarn or cables deployed off of spools or onto the seafloor (Mack, 1958; Zajac, 1957; Padfield, 1958; Mankala and Agrawal, 2005; Krupa et al., 2006), pipes conveying fluid (Païdoussis and Li, 1993), or other situations in which one or more boundaries act as sources or sinks of material.

This paper is restricted to metrically constrained inertial motions in $\mathbb{E}^3$, but the method can be generalized to other systems involving elasticity or plasticity of the body. Aside from the economy of assumptions inherent to a variational principle, there seem to be conceptual advantages to viewing a discontinuity as a moving boundary, rather than an internal "wave". There is the possibility of treating fracture, as well as combined line and point defects, by the approach suggested in Section 4. The ability to consider boundary conditions of action functionals of arbitrarily high derivatives of position is exploited in Section 6 to derive momentum and energy balances for a discontinuous *elastica*. Finally, the present treatment adds a new perspective on the existence of energy functionals for some axial motions of thin bodies between supports (Renshaw et al., 1998).

The prior work of McIver (1973) should be mentioned, in which an action principle was developed for an open system akin to the time-dependent volumes considered in the present work. The present treatment differs primarily in its focus on boundary conditions and their application to discontinuities, and in its use of global material body coordinate descriptions of the moving boundaries.

The variations in the current procedure involve only the material position vector and the time; it is likely that these could be viewed as a single four-dimensional material position vector. This is in contrast to variations of the position of geometric quantities, such as the location of a boundary or other defect. Examples of the latter may be found in recent treatments of static adhesion (Deserno et al., 2007; Majidi and Adams, 2009; Majidi et al., 2012; Hure and Audoly, 2013), and in approaches based on the concept of configurational balances (Gurtin, 2000; Kienzler and Herrmann, 2000; Maugin, 2011). The current procedure is more direct, not requiring any additional compatibility conditions on the variations. Perhaps more importantly, it does not rely on any principles beyond the established action of classical mechanics that applies to the material composing the body; there are no new postulated laws for geometric objects. However, the approach is limited to defects that can be described as boundaries, excluding non-Riemannian objects such as dislocations.