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Energy change to insertion of inclusions associated with the Reissner–Mindlin plate bending model

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ABSTRACT

The topological derivative concept has been proved to be useful in many relevant applications such as topology optimization, inverse problems, image processing, multi-scale constitutive modeling, fracture mechanics and damage evolution modeling. The topological asymptotic analysis has been fully developed for a wide range of problems modeled by partial differential equations. On the other hand, the topological derivatives associated with coupled problems have been derived only in their abstract forms. In this paper, therefore, we deal with the Reissner–Mindlin plate bending model, which is written in the form of a coupled system of partial differential equations. In particular, the topological asymptotic analysis of the associated total potential energy is developed and the topological derivative with respect to the nucleation of a circular inclusion is derived in its closed form. Finally, we provide the estimates for the remainders of the topological asymptotic expansion and perform a complete mathematical justification for the derived formulas.

1. Introduction

The topological asymptotic analysis leads to the asymptotic expansion of a given shape functional with respect to a singular domain perturbation such as holes, inclusions, cracks, etc. The main term of such expansion is defined as the topological derivative (Sokołowski and Żochowski, 1999), which has been proved to be useful in many relevant applications such as topology optimization (Amstutz et al., 2012), inverse problems (Hintermüller et al., 2012), image processing (Hintermüller and Laurain, 2009), multi-scale constitutive modeling (Amstutz et al., 2010), fracture mechanics (Van Goethem and Novotny, 2010) and damage evolution modeling (Allaire et al., 2011). For a comprehensive account on the topological derivative concept see, for instance, the book by Novotny and Sokołowski (2013).

In particular, the topological derivative for the Kirchhoff plate bending problem has been rigorously derived by Amstutz and Novotny (2011), which involves a forth-order differential operator. For the topological asymptotic analysis associated with higher-order elliptic differential operators see Amstutz et al. (2014). On the other hand, the topological derivative associated with the Reissner–Mindlin plate bending problem has not been reported in the literature yet. This mechanical model leads to a coupled system of second-order partial differential equations. In fact, only a few works dealing with coupled problems can be found in the literature, whose derived formulas are presented only in their abstract forms (Cardone et al., 2010).

Therefore, in this work the topological derivative for the total potential energy associated with the Reissner–Mindlin plate bending problem is derived. In particular, arguments on the existence of the topological derivative for this model are presented, together with precise estimates for the remainders of the associated topological asymptotic expansion. Finally, the topological derivative with respect to the nucleation of an infinitesimal circular inclusion is derived in its closed form. Since the Reissner–Mindlin plate bending model takes into account the shear effects, we believe that the derived formula shall be useful for practical applications, allowing for overcome some numerical difficulties associated with the Kirchhoff plate bending model reported by Novotny et al. (2005), for instance.

This paper is organized as follows. In Section 2 we introduce the topological derivative concept. The mechanical model which we are dealing with is presented in Section 3, together with the existence of the associated topological derivative. The explicit form of the topological derivative is derived in Section 4. The estimates for the remainders of the topological asymptotic expansion are presented in Section 5. Finally, the paper ends with some concluding remarks in Section 6.

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2. The topological derivative concept

Let us consider an open and bounded domain $\Omega \subset \mathbb{R}^2$, which is subject to a nonsmooth perturbation confined in a small region $\omega_\varepsilon(\widehat{x}) = \widehat{x} + \varepsilon\omega$ of size ε and center at $\widehat{x} \in \Omega$, as shown in Fig. 1.

We introduce a characteristic function associated to the unperturbed domain $x \mapsto \chi(x), x \in \mathbb{R}^2$, of the form $\chi = \mathbb{1}_\Omega$, so that:

$$|\Omega| = \int_{\mathbb{R}^2} \chi. \tag{1}$$

On the other hand, we define a piecewise constant function associated to the perturbed domain $x \mapsto \chi_\varepsilon(\widehat{x}, x), x \in \mathbb{R}^2$, with ε and $\widehat{x} \in \Omega$ fixed, so that $\chi_\varepsilon(\widehat{x}) = \mathbb{1}_\Omega - (1-\gamma)\mathbb{1}_{\omega_\varepsilon(\widehat{x})}$. Therefore,

$$\chi_\varepsilon(\widehat{x}, x) = \begin{cases} 1 & \text{if} \quad x \in \Omega \setminus \omega_\varepsilon(\widehat{x}), \\ \gamma & \text{if} \quad x \in \omega_\varepsilon(\widehat{x}), \\ 0 & \text{if} \quad x \in \mathbb{R}^2 \setminus \Omega, \end{cases} \tag{2}$$

where $0 < \gamma < \infty$ is the contrast on the material properties.

Now, we assume that a given shape functional associated to the topological perturbed problem $\psi(\chi_\varepsilon(\widehat{x}))$ admits the following topological asymptotic expansion:

$$\psi(\chi_\varepsilon(\widehat{x})) = \psi(\chi) + f(\varepsilon)\mathcal{T}(\widehat{x}) + o(f(\varepsilon)), \tag{3}$$

where $\psi(\chi)$ is the shape functional associated to the unperturbed problem, $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$, when $\varepsilon \to 0^+$. Function $\widehat{x} \mapsto \mathcal{T}(\widehat{x})$ is defined as the topological derivative of ψ at the point $\widehat{x}$. This derivative can be understood as a first order correction on $\psi(\chi)$ to approximate $\psi(\chi_\varepsilon(\widehat{x}))$. After rearranging the Eq. (3), the limit passage $\varepsilon \to 0^+$ leads to:

$$\mathcal{T}(\widehat{x}) = \lim_{\varepsilon \to 0^+} \frac{\psi(\chi_\varepsilon(\widehat{x})) - \psi(\chi)}{f(\varepsilon)}. \tag{4}$$

Note that in our particular case the geometrical domains remain fixed, while the topological perturbation is going to be driven by a contrast γ on the material properties. In the singular case associated with the nucleation of holes, e.g. $\gamma = 0$, the analysis becomes much more involved and can be seen in details in Nazarov et al. (2010), for instance.

3. Thick plate bending model

In this section we introduce a plate bending problem under the kinematic assumptions of Reissner–Mindlin (Mindlin, 1951; Reissner, 1945). Thus, let us consider a plate represented by a two-dimensional domain $\Omega \subset \mathbb{R}^2$, with thickness $h > 0$ supposed to be constant for the sake of simplicity. We assume that the plate is submitted to bending and shear effects under the following kinematic assumptions:

> The normal fibers to the middle plane of the plate remain straight during the deformation process, but not necessarily normal to the middle plane, and do not suffer variations in their length. Consequently, the transversal shear deformations are not negligible and the normal deformations are null.

The Reissner–Mindlin plate bending problem leads to a coupled system of second-order partial differential equations, which is known to be strongly elliptic. See, for instance (Rössle and Sändig, 2011; McLean, 2000, Ch. 4). This important property will be exhaustively used in the derivations to be presented in what follows.

3.1. Unperturbed problem

Let us introduce the total potential energy associated with the unperturbed plate problem, namely:

$$\psi(\chi) := \mathcal{J}(\theta, w) = \frac{1}{2}\int_\Omega (\mathcal{M}(\theta)\cdot\nabla^s\theta + \mathcal{Q}(\theta, w)\cdot(\theta - \nabla w)) - \int_{\Gamma_{N_\theta}} \overline{m}\cdot\theta + \int_{\Gamma_{N_w}} \overline{q}w, \tag{5}$$

where $\mathcal{M}(\theta) = \mathbb{C}\nabla^s\theta$ is the generalized bending moment tensor and $\mathcal{Q}(\theta, w) = \mathbf{K}(\theta - \nabla w)$ is the generalized shear tensor. The constitutive fourth $\mathbb{C}$ and second $\mathbf{K}$ order tensors are assumed to be isotropic and homogeneous, which are respectively given by

$$\mathbb{C} = \frac{Eh^3}{12(1-\nu^2)}((1-\nu)\mathbb{I} + \nu(\mathbf{I}\otimes\mathbf{I})), \tag{6}$$

$$\mathbf{K} = \frac{\kappa Eh}{2(1+\nu)}\mathbf{I}, \tag{7}$$

where E is the Young modulus, ν is the Poisson ration, $\kappa = 5/6$ is the shear correction factor and h the plate thickness. In addition, $\mathbf{I}$ and $\mathbb{I}$ are the second and fourth order identity tensors, respectively. The rotation θ and the transversal displacement w are solutions to the following coupled variational problem: For all $(\eta_\theta, \eta_w) \in \mathcal{V}$, find the field $(\theta, w) \in \mathcal{U}$, such that

$$\begin{cases} \int_\Omega (\mathcal{M}(\theta)\cdot\nabla^s\eta_\theta + \mathcal{Q}(\theta, w)\cdot\eta_\theta) = \int_{\Gamma_{N_\theta}} \overline{m}\cdot\eta_\theta, \\ \int_\Omega \mathcal{Q}(\theta, w)\cdot\nabla\eta_w = \int_{\Gamma_{N_w}} \overline{q}\eta_w. \end{cases} \tag{8}$$

In the variational problem (8), the set $\mathcal{U}$ of admissible functions and the space $\mathcal{V}$ of admissible variations are defined by:

$$\mathcal{U} := \left\{(\varphi_\theta, \varphi_w), \varphi_\theta \in \mathbf{H}^1(\Omega) \text{ and } \varphi_w \in H^1(\Omega) : \varphi_w|_{\Gamma_{D_w}} = \overline{w}, \varphi_\theta|_{\Gamma_{D_\theta}} = \overline{\theta}\right\}, \tag{9}$$

$$\mathcal{V} := \left\{(\varphi_\theta, \varphi_w), \varphi_\theta \in \mathbf{H}^1(\Omega) \text{ and } \varphi_w \in H^1(\Omega) : \varphi_w|_{\Gamma_{D_w}} = 0, \varphi_\theta|_{\Gamma_{D_\theta}} = 0\right\}, \tag{10}$$

with $\mathbf{H}^1(\Omega) := H^1(\Omega)\times H^1(\Omega)$, where Γ_{N_θ} and Γ_{N_w} are Neumann boundaries, while Γ_{D_θ} and Γ_{D_w} are Dirichlet boundaries. Then, $\overline{\theta} \in \mathbf{H}^{\frac{1}{2}}(\Gamma_{D_\theta})$ is a Dirichlet data representing a rotation prescribed on Γ_{D_θ}, while $\overline{w} \in H^{\frac{1}{2}}(\Gamma_{D_w})$ is a Dirichlet data associated with a transversal displacement prescribed on Γ_{D_w}. In addition, the Neumann data are given by $\overline{m} \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_{N_\theta})$, representing a distributed moment prescribed on Γ_{N_θ}, and $\overline{q} \in H^{-\frac{1}{2}}(\Gamma_{N_w})$, representing a distributed shear on Γ_{N_w}. Finally, $\Gamma_{D_w} \cap \Gamma_{N_w} = \emptyset$ and $\Gamma_{D_\theta} \cap \Gamma_{N_\theta} = \emptyset$,

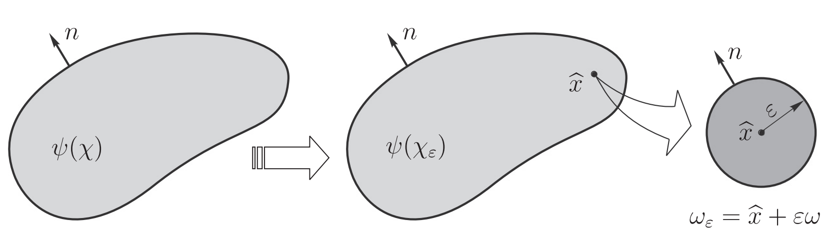

Fig. 1. The topological derivative concept.