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A higher-order strain gradient plasticity theory with a corner-like effect

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ABSTRACT

A corner-like plasticity model originally proposed as a size-independent theory is extended to include a size effect resulting from plastic strain gradients. A method of solving boundary value problems at finite strains is also presented. The efficiency of the new theory is demonstrated through two typical numerical examples: a constrained simple shear problem in which an infinitely long strip bounded by two hard materials is subjected to large shear under plane strain conditions, and a problem of shear band formation in plane strain tension.

1. Introduction

Whether or not a vertex (or a corner) develops on the yield surface of a plasticity material is a very important inquiry in predictions of plastic instability that accompanies an abrupt change in the deformation mode without a large variation in the stress state (Støren and Rice, 1975; Hutchinson and Tvergaard, 1981). Plasticity theories accounting for the effects of a yield surface corner have been proposed by Christoffersen and Hutchinson (1979) and Gotoh (1985). Later, for the purpose of developing simplified and efficient numerical procedures, Hughes and Shakib (1986) proposed a pseudo-corner theory representing the reduced stiffness and increased plastic flow of the corner-theory-like response to non-proportional loading. Simo (1987) proposed a non-associative flow rule that represents a corner-like effect on an apparently smooth yield surface.

An experimental investigation on the use of an abrupt strain path change to determine the shape of a subsequent yield surface in the vicinity of a current loading point was carried out by Kuwabara et al. (2000), which was based on a method proposed by Kuroda and Tvergaard (1999). The recorded trajectory of stress immediately after a strain path change, which inevitably travels close to the current yield surface, strongly indicated the existence of a yield surface vertex at the loading point. Furthermore, the direction of the plastic strain rate was clearly inclined toward the forward direction of the stress path that is regarded as part of the section of the current yield surface. That is, a large deviation of the plastic strain rate direction from the normal to the apparent yield surface was observed. This is interpreted to mean that when the stress point moves along what appears to be a smooth yield surface, a vertex moves with the stress point, which explains the apparent nonnormality. These experimental observations are similar to the numerical predictions obtained using the Taylor polycrystal model (Kuroda and Tvergaard, 1999). The observed apparent nonnormality effect on a smooth yield surface is consistent with the basic idea of Simo's corner-like plasticity model (Simo, 1987). Kuroda and Tvergaard (2001a) modified Simo's model so that it can represent observations in the polycrystal model (Kuroda and Tvergaard, 1999) and the experiments of Kuwabara et al. (2000) as closely as possible at a macroscopic level of modeling. The modified model of Kuroda and Tvergaard (2001a) was applied to finite element analysis to predict the shear band development in plane strain tensile specimens, and it was confirmed that the results were reasonably close to crystal plasticity predictions (Kuroda and Tvergaard, 2001c).

Conventional plasticity theories including the aforementioned classes of corner theories do not account for any size effect in real material behavior. A wide array of experiments on micron-size specimens have revealed significant size-dependent mechanical behaviors in plastically strained materials involving spatial gradients of strain (e.g., Fleck et al., 1994; Stölken and Evans, 1998). On the theoretical side, a considerable number of studies have been conducted with the aim of incorporating the strain gradient effects into theories of plasticity since the pioneering studies of Aifantis (1984, 1987). Part of them were carried out within a context of the crystal plasticity framework (e.g., Gurtin, 2002; Evers et al., 2004; Kuroda and Tvergaard, 2006; Borg, 2007). However, in parallel, extensions of phenomenological plasticity theories (in most cases, J_2-based theories) have attracted considerable

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attention owing to their simplicity and practical efficiency (e.g., Fleck et al., 1994; Fleck and Hutchinson, 2001; Gudmundson, 2004; Gurtin and Anand, 2009; Hutchinson, 2012). The former approach is advantageous for performing physically based simulations directly accounting for the effects of the geometrically necessary dislocations (GNDs), which correspond to gradients of crystallographic slips (Ashby, 1970). The latter approach is suitable in cases where one intends to model a resultant size effect in polycrystalline materials or introduce an assumption of isotropy, as a first approximation, for micron-scale plasticity. It is now widely recognized that these gradient plasticity theories must be *higher-order* in the sense that it should be possible to impose extra boundary conditions with respect to plastic strains or their gradients, which are outside the scope of conventional plasticity theories.

In the present study, the corner-like plasticity model of Kuroda and Tvergaard (2001a), whose original version was proposed by Simo (1987), is extended to include a size effect resulting from plastic strain gradients (Aifantis, 1984, 1987). Then, a method of solving boundary value problems at finite strains using the proposed constitutive model is presented. Two typical examples are shown to demonstrate the efficiency of the new theory: one is a constrained simple shear problem in which an infinitely long strip bounded by two hard materials is subjected to large shear under plane strain conditions, and the other is a problem of shear band formation in plane strain tension.

2. Theory and solution method

2.1. Constitutive modeling

Let $\mathbf{x}$ be the current position of a particle labeled $\mathbf{X}$ in the undeformed configuration of the body under consideration, $\mathbf{F} = \partial\mathbf{x}/\partial\mathbf{X}$ be the deformation gradient, $\mathbf{L} = \dot{\mathbf{F}}\cdot\mathbf{F}^{-1} = \partial\dot{\mathbf{x}}/\partial\mathbf{x} = \partial\dot{\mathbf{u}}/\partial\mathbf{x} = \dot{\mathbf{u}}\otimes\nabla$ be the velocity gradient (where $\mathbf{u}$ is the displacement, a superposed dot denotes a material-time derivative, ∇ is the spatial gradient operator with respect to the current configuration, and $\otimes$ is the tensor product operator), $\mathbf{D} = \frac{1}{2}(\mathbf{L}+\mathbf{L}^{\mathrm{T}})$ be the deformation rate tensor, and $\mathbf{W} = \frac{1}{2}(\mathbf{L}-\mathbf{L}^{\mathrm{T}})$ be the continuum spin tensor. An additive decomposition of the deformation rate tensor into elastic and plastic parts is postulated as $\mathbf{D}^{\mathrm{e}} = \mathbf{D} - \mathbf{D}^{\mathrm{p}}$. The elastic response is assumed to be given by the following hypoelasticity relation

$$\overset{\circ}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \mathbf{W}\cdot\boldsymbol{\sigma} + \boldsymbol{\sigma}\cdot\mathbf{W} = \mathbf{C} : \mathbf{D}^{\mathrm{e}} \tag{1}$$

with

$$\mathbf{C} = \lambda\mathbf{1}\otimes\mathbf{1} + 2\mu\mathbf{I}^{(4s)}, \tag{2}$$

where $\boldsymbol{\sigma}$ is the Cauchy (true) stress, $\overset{\circ}{\boldsymbol{\sigma}}$ is its Jaumann rate, $\mathbf{C}$ is a fourth-order elastic moduli tensor, λ and μ are Lamé constants, $\mathbf{1}$ is the second-order identity tensor, and $\mathbf{I}^{(4s)}$ is the fourth-order symmetric identity tensor. For plastic deformation, the following form of the flow rule is adopted:

$$\mathbf{D}^{\mathrm{p}} = \dot{\phi}\mathbf{N}^{\mathrm{p}}, \tag{3}$$

where $\dot{\phi}$ is a scalar plastic multiplier and the tensor $\mathbf{N}^{\mathrm{p}}$ represents the direction of $\mathbf{D}^{\mathrm{p}}$, which will be defined later.

A gradient-dependent yield condition is introduced, following Aifantis (1984), as

$$f = \sigma_{\mathrm{e}} + \nabla\cdot\mathbf{g}^{\mathrm{p}} - R(\varepsilon^{\mathrm{p}}) = 0 \tag{4}$$

for a plastic loading condition, and $f < 0$ for elastic states (i.e. before initial yielding or in an unloaded state), where

$$\sigma_{\mathrm{e}} = \sqrt{\frac{3}{2}}|\boldsymbol{\sigma}'|; \quad \boldsymbol{\sigma}' = \boldsymbol{\sigma} - \frac{1}{3}\boldsymbol{\sigma} : (\mathbf{1}\otimes\mathbf{1}), \tag{5}$$

$$\mathbf{g}^{\mathrm{p}} = \beta\nabla\varepsilon^{\mathrm{p}}, \tag{6}$$

$R(\varepsilon^{\mathrm{p}})$ is a function of the equivalent plastic strain ε^{p}, which represents a work-hardened state of the material, β is a length scale coefficient assumed to be constant in the present study, a prime, $(\bullet)'$, denotes the deviatoric part of the tensor $(\bullet)$, and $|(\bullet)| = \sqrt{\mathrm{tr}[(\bullet)^{\mathrm{T}}\cdot(\bullet)]}$.

The introduction of the gradient term into yield conditions was mainly motivated by the desire to predict the postlocalization features of material behavior. The inclusion of the gradient term is necessary to determine the shear band width in the post-bifurcation regime as first discussed by Aifantis (1984, 1987). Although more complicated or generalized strain gradient formulations have been proposed by different researchers (e.g., Fleck and Hutchinson, 2001; Gudmundson, 2004), Eq. (4) is used here as the first choice. Eq. (4) is the simplest, but involves the primary effect of the plastic strain gradients. A physical basis for the introduction of $\nabla\cdot\mathbf{g}^{\mathrm{p}}(=\beta\nabla^2\varepsilon^{\mathrm{p}}$ for a constant $\beta)$ can be strengthened by an argument based on the dislocation theory, as discussed by Kuroda and Tvergaard (2006, 2010). That is, a dislocation-induced long-range internal stress arises in response to spatial gradients of the GND density (Groma et al., 2003; Evers et al., 2004), and the GND density is equated with the spatial gradient of crystallographic slip (Ashby, 1970). Thus, the dislocation-induced internal stresses correspond to the second gradients of crystallographic slips. The macroscopic yield condition with the term $\beta\nabla^2\varepsilon^{\mathrm{p}}$ is mathematically similar to microscopic yield conditions for each slip system in the gradient crystal plasticity theories (Groma et al., 2003; Evers et al., 2004). Based on this view, we can identify the term $\beta\nabla^2\varepsilon^{\mathrm{p}}$ with a resultant of the GND density-induced internal stress. On this understanding, *plastic dissipation* should be accounted for by $R\dot{\varepsilon}^{\mathrm{p}}(\geqslant 0)$ as pointed out by Kuroda and Tvergaard (2010).

The direction tensor $\mathbf{N}^{\mathrm{p}}$ in Eq. (3) is taken to be

$$\mathbf{N}^{\mathrm{p}} = \underset{\sim}{\mathbf{n}} + \hat{\delta}\,\underset{\sim}{\mathbf{m}} \tag{7}$$

with

$$\underset{\sim}{\mathbf{n}} = \frac{\partial f/\partial\boldsymbol{\sigma}}{|\partial f/\partial\boldsymbol{\sigma}|} = \frac{\boldsymbol{\sigma}'}{|\boldsymbol{\sigma}'|}, \quad \underset{\sim}{\mathbf{m}} = \frac{\mathbf{D}' - (\underset{\sim}{\mathbf{n}} : \mathbf{D}')\underset{\sim}{\mathbf{n}}}{|\mathbf{D}' - (\underset{\sim}{\mathbf{n}} : \mathbf{D}')\underset{\sim}{\mathbf{n}}|} \tag{8}$$

The framework of this flow rule with a corner-like effect was originally proposed by Simo (1987). In the original model, $\hat{\delta}$ was formulated so that $\mathbf{D}^{\mathrm{p}}$ is always coaxial to $\mathbf{D}'$ when $\mathbf{D}'$ lies inside a hypercone defined by semi-angle $\Theta^{\mathrm{p}}_{\mathrm{crit}}$ (measured from the direction $\underset{\sim}{\mathbf{n}}$; $\mathbf{D}^{\mathrm{p}}$ must lie within the hypercone). Later, Kuroda and Tvergaard (2001a) modified the relation for $\hat{\delta}$ to

$$\hat{\delta} = \tan\Theta^{\mathrm{p}}; \qquad \Theta^{\mathrm{p}} = \begin{cases} a\Theta & \text{for} \quad a\Theta \leqslant \Theta^{\mathrm{p}}_{\mathrm{crit}} \\ \Theta^{\mathrm{p}}_{\mathrm{crit}} & \text{for} \quad a\Theta > \Theta^{\mathrm{p}}_{\mathrm{crit}} \end{cases}, \tag{9}$$

$$\Theta = \cos^{-1}\left[\frac{\underset{\sim}{\mathbf{n}} : \mathbf{D}'}{|\mathbf{D}'|}\right], \quad a = \left(c\frac{R(\varepsilon^{\mathrm{p}})}{\mu} + 1\right)^{-1}, \tag{10}$$

where c is a coefficient that introduces non-coaxiality between the total deviatoric strain rate $\mathbf{D}'$ and the plastic strain rate $\mathbf{D}^{\mathrm{p}}$. A schematic diagram of the present corner-like model is shown in Fig. 1. The introduction of non-coaxiality between $\mathbf{D}'$ and $\mathbf{D}^{\mathrm{p}}$ is a necessity to reproduce the corner effect observed in crystal plasticity. Kuroda and Tvergaard (2001a) suggested that a range of c from 3 to 10 might be realistic through identification using Taylor model computations involving an abrupt strain path change. In a subsequent study on shear band simulations (Kuroda and Tvergaard, 2001c), it was found that a slightly smaller value, $c = 2$, was most suitable for reproducing the shear band development behavior predicted in crystal plasticity simulations. The present corner-like model