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A nonlinear integral model for describing responses of viscoelastic solids

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ABSTRACT

In this paper we develop a model as well as carry out experiments to test the efficacy of the model, for a class of non-aging isotropic viscoelastic solids. We fashion a nonlinear integral model, which belongs to the class of quasi-linear viscoelastic models, for solid-like materials, which upon linearization reduces to a linear viscoelastic model. The model is defined by separating the normalized time function and nonlinear stress measure that describes the elastic response of the materials. In the case of isotropic materials we consider two independent normalized time functions and two nonlinear stress measures, which are expressed in terms of the stress invariants I_1 and I_2. We discuss the methodology of the material parameter characterization based on the experimental data available for polyoxymethylene (POM) under quasi-static ramp with constant rate and creep loadings. The response predicted by the nonlinear integral model calibrated by the material parameters obtained through data reduction is then tested against other experimental data under various loading histories. The nonlinear viscoelastic model is capable of capturing the three-dimensional response of POM polymers under various histories of inputs both under stress and strain controlled loadings. Finally we present solutions to boundary value problems (BVPs) of a concentric cylinder made of POM under internal pressure in order to demonstrate one of the applications in which POM is used. The design of structural elements made of POM can be undertaken by using models of the class developed in this work.

1. Introduction

Many materials, especially polymers and biological tissues can be best described as viscoelastic bodies in that they can store energy like elastic solids and dissipate energy like viscous fluids, which at the macroscopic scale are reflected in such bodies exhibiting stress relaxation and creep; that is time-dependent response when subject to mechanical stimulus. Various types of constitutive models, which can be broadly categorized under differential, rate type and integral type models have been developed for describing the viscoelastic responses of materials. Both linear and non-linear models have been used to describe the response of viscoelastic bodies and the most popular model is the linearized model wherein the gradient of the displacement and hence the strain is assumed to be small and an expression for the stress is provided in terms of the linearized strain or an expression for the linearized strain is provided in terms of the history of the stress. In this paper, we are primarily interested in formulating a nonlinear viscoelastic integral model for solid-like materials, which upon linearization reduces to a linear viscoelastic model. Furthermore, we restrict the formulation to non-aging and isotropic viscoelastic bodies.

In a linear viscoelastic body the expression for the time-dependent stress and strain is interchangeable, that is either the stress can be expressed in terms of the history of the linearized strain or one can invert the expression in order to obtain the expression for the linearized strain in terms the history of the stress. The responses of a linear viscoelastic body to a sum of inputs can be obtained by the superposition of the responses to the individual inputs. The linear viscoelastic[1] model is only applicable when the body undergoes small displacement gradients. It is however possible for a viscoelastic body to undergo small displacement gradients, in which a linearized strain measure is sufficient to describe the deformation of the body; however, its responses may not meet the proportionality and superposition of response that a linear viscoelastic body meets whereby one needs to recognize it as a nonlinear viscoelastic body. In such a situation, it is reasonable to express

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[1] When we refer to a linear viscoelastic body, we deal with two sources of linearity. First, we consider only the linear term of the displacement gradients in our strain measures, leading to the linearized strain. Second, the responses are proportional to the inputs and can be obtained by superimposing the responses of several different inputs.

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the linearized strain, $\boldsymbol{\varepsilon}$, as a function of the stress, $\boldsymbol{\sigma}$, history $\boldsymbol{\varepsilon}(t) = F[\boldsymbol{\sigma}(t-s)]$, instead of vice versa. The rationale for such a representation is a consequence of the fact that the stress does not have to be small, but the expression for the linearized strain in terms of the stress leads to a sufficiently small value for the strain. The stress cannot be in general expressed in terms of a nonlinear function of the linearized strain, $\boldsymbol{\sigma}(t) = G[\boldsymbol{\varepsilon}(t-s)]$, there is the limitation with regard to the sort of function G that can be chosen; it cannot involve any higher order terms of $\boldsymbol{\varepsilon}$. Moreover, from the point of view of causality (see Truesdell, 1966; Rajagopal, 2013), as force is applied to a body to cause it to undergo deformation, that is force is the cause and the deformation is the effect, it makes eminent sense to express the effect in terms of the cause rather than vice versa. From a practical stand point, it is relatively easier to perform creep tests than relaxation tests, so that the material parameters in the constitutive models that are expressed in terms of the applied stress can be easily calibrated from the experimental data.[2]

There have been several integral models developed for describing the nonlinear viscoelastic response of materials. Reviews of the nonlinear viscoelastic models that are currently used can be found in Drapaca et al. (2007) and Wineman (2009). Green and Rivlin (1957) and Coleman and Noll (1961) developed constitutive models for materials with fading memory, which reflect the fact that the response of materials at the present time depends more strongly on the deformations which occurred in the recent past than those that occurred in the distant past. Both Green and Rivlin (1957) and Coleman and Noll (1961) expressed their constitutive models for the stress in terms of the history of strain. It was assumed that the dependence of the stress on the history of strain should be smooth and continuous at all times, which allows using a Taylor series expansion in order to express the time-dependent constitutive models in terms of the strain history. This expression leads to multiple integral constitutive models for nonlinear viscoelastic materials. The time-dependent kernels in the integrals must be positive, continuous, and monotonically decreasing functions of time in order to describe the stress relaxation in the materials. It is noted that the strain history, as input, does not have to be continuous. When the displacement gradients are relatively small, the multiple integral models reduce to a linear viscoelastic integral model (only the first term is sufficient to describe the responses of the materials). Pipkin and Rogers (1968) expressed the time-dependent stress in terms of the history of strain rate rather than the history of strain. They further discussed the possibility that such integral expressions are also valid when the roles of stress and strain are interchanged.[3] They showed that for materials with strong nonlinearity, e.g. polyurethane foams, it is necessary to include higher order multiple integrals, in which they considered third order multiple integral representation to model creep deformation of polyurethane foam. The inclusions of higher order terms often lead to difficulties with regard to experimental data reduction, i.e., multiple step tests are required to characterize the time kernels in a one dimensional multiple integral model. Pipkin and Rogers (1968) later suggested that in order to describe the response of materials with strong nonlinearity, the first term of the integral with nonlinear integrand can be used, resulting in a nonlinear single integral model. This model is based on the modified superposition method.

Fung (1981) proposed a quasi-linear viscoelastic (QLV) model in order to describe the viscoelastic response of biological materials. In the QLV model, the stress relaxation function is modeled by the separation of two functions, which are the reduced (normalized) time functions,[4] $K(t)$, and nonlinear elastic function $F^{el}[\mathbf{E}(s)]$, where $\mathbf{E}(s)$ is the Green–St. Venant strain and it is necessary that $K(0) = 1$. The nonlinear elastic function $F^{el}[\mathbf{E}(s)]$ can be derived from the strain energy density function (see Dai et al., 1992; Johnson et al., 1996; Puso and Weiss, 1998). The QLV model has both mathematical and experimental advantages since it reduces the complexity in solving the constitutive equations and material parameter characterizations. The reduced time function (or reduced relaxation function) is not unique and any function that is positive, continuous, and monotonically decreasing with time is acceptable. Pipkin (1986) and Wineman and Rajagopal (2000) discussed several relaxation functions, either based on a discrete or continuous relaxation spectrum. Schapery (1969) formulated a nonlinear single integral model for describing the nonlinear viscoelastic response of polymers undergoing small deformations. There he introduced four nonlinear parameters associated with the instantaneous (elastic), transient, loading rate, and accelerated/decelerated time-dependent responses. He also discussed dual representations, where the roles of stress and strain are interchanged. Muliana et al. (2013) used a QLV model in order to describe the nonlinear response of viscoelastic materials in which the strain remains relatively small. They expressed the linearized strain in terms of the history of a nonlinear measure of the stress and presented one dimensional analyses to simulate the response of a human patellar tendon subject to a relatively small uniaxial strain (less than 5%). Recently, De Pascalis et al. (2014) have discussed the derivation of Fung's QLV model starting from the basic nonlinear kinematics, which guarantees consistency with a linear viscoelastic behavior when the deformation gradient is small, and objectivity in that the QLV is expressed in terms of the second Piola–Kirchhoff stress tensor. They presented the QLV model explicitly in terms of the strains and discussed that when the deformation is imposed the corresponding stresses can be directly determined.

Many constitutive models in elasticity and viscoelasticity are expressed in explicit forms of the strain, i.e., the stress is expressed in terms of the strain. One of the reasons behind this popular form is perhaps due to the simplicity that is a consequence of such an assumption in solving the governing equations to boundary value problems as the governing equations can be expressed explicitly in terms of the displacement fields. Once the displacement fields are determined, the stress field can be directly calculated from the constitutive relations. To further reduce mathematical complexity in finding the exact solutions, incompressible material behavior is often assumed. When the strain (or displacement) is explicitly expressed in terms of the stress, mathematical complexity might arise since it might be necessary to simultaneously solve the balance equations and the constitutive equations. However, with regard to the material parameter characterizations from experiments, we have a straight forward characterization when the stress history is prescribed for the body. We will discuss this issue later in Section 4 of this manuscript.

In this paper, we formulate a quasi-linear viscoelastic model for non-aging isotropic materials. Consistent with the issue of causality and practical material characterization from stress-controlled tests, we express the time-dependent strain in terms of the history of stress. We also present procedures to calibrate the three-dimensional nonlinear stress measures and time-dependent functions from experimental data. Uniaxial tensile tests on polyoxymethylene (POM) under various loading histories are conducted, and

[2] For example, Pipkin and Rogers (1968) discussed their nonlinear viscoelastic model in general by considering the stress in terms of the history of the strain. They also presented material parameter characterization and data reduction methods from the creep tests with multiple steps, in which they used the expression for strain in terms of the history of the stress in order to calibrate the material parameters.

[3] The time-dependent kernel function related to creep response must be positive, continuous and an increasing function of time.

[4] In a three-dimensional representation, one can use a reduced time fourth order tensor.