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Embedded homogeneity of beams in the nonlinear domain

Marcelo Epstein [a,*], Ayan Roychowdhury [b]

[a] Department of Mechanical and Manufacturing Engineering, University of Calgary, Canada
[b] Department of Mechanical Engineering, Indian Institute of Technology Kanpur, India

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ABSTRACT

The notion of embedded homogeneity of thin-walled structures is introduced as the property characterizing the provenance of such a structure from a homogeneous material. This property needs to be distinguished from other definitions of homogeneity formulated exclusively in terms of a purely structural constitutive equation. Necessary conditions for embedded homogeneity are derived for planar beams and their geometric interpretation is expressed as the condition for the elastic hodograph to lie on a hypersphere containing the origin of a six-dimensional space of tensors.

1. Introduction

An elastic material body is said to be *materially uniform* if all its points are made of the same material. A materially uniform body is *(globally) homogeneous* if it can be brought to a configuration in which, when expressed in a system of Cartesian coordinates, the constitutive equation is independent of position. The precise mathematical statement of these conditions is well established within the general framework of Continuum Mechanics, most particularly in Noll (1967), Wang (1967). On the other hand, the notions of material uniformity and homogeneity of thin-walled structures (beams and shells) are less definitely understood due to the intertwining between the material properties and the geometry of the body. This interesting interplay has been approached from various points of view by different authors (e.g., Wang, 1972; Cohen and Epstein, 1983; Naghdi and Rubin, 1995; Epstein and de León, 1998). As befits a structural theory, the question of homogeneity is framed within the context of certain properties of the structural constitutive equations. Thus, the constitutive equation of a plane beam is expressed, say, in terms of a strain energy that depends on the axial elongation and the change of curvature of the beam axis. A legitimate question that can be asked, however, is whether the beam *as a bona fide three dimensional material body* is homogeneous. More dramatically, one may ask whether a material scientist looking at a small piece of material extracted from a beam can recognize anything but a piece of material and, moreover, whether this material element contains distributed dislocations. This way of formulating the question of homogeneity of a thin-walled structure has led us to define the notion of *embedded homogeneity*. We say that a thin-walled structure enjoys this property if it can be conceived as having been cut from a homogeneous material block. When this is the case, it is clear that the derived constitutive equation of the thin-walled structure can be obtained by a process of integration over the thickness of the structural element via certain kinematic assumptions (such as the preservation of the normal to the middle surface of a shell). What interests us here is the inverse problem, namely, given a constitutive equation of a thin-walled structure, can it be so derived from an extension to an ordinary material body? In a previous article, Epstein and Roychowdhury (2014), we have tackled this question within the limited framework of a linearized elasticity theory. The results were expressed analytically and then interpreted geometrically in an unexpectedly elegant way. In the present work we show, by a different approach, that the conditions of embedded homogeneity derived in Epstein and Roychowdhury (2014) are valid also as necessary conditions in the nonlinear elastic range. The presentation in this paper is self-contained and, somewhat paradoxically, less involved than that in Epstein and Roychowdhury (2014).

2. Intuitive considerations

Let the homogeneity of a two-dimensional body be, naively perhaps, identified with the existence of a stress-free configuration in which the "atoms" are arranged in a perfectly regular orthogonal lattice (Fig. 1) within a material block. If a curved strip were to be carved out of this block, the result would be, again, a homogeneous body with the geometric appearance of what we habitually

* Corresponding author. Tel.: +1 403 2205791.
E-mail address: mepstein@ucalgary.ca (M. Epstein).

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call a beam. While there is no doubt that, as a two-dimensional entity, this body is homogeneous, we ask the question: Is this a homogeneous *beam*? Clearly, if, for example, the original block is made of an orthotropic elastic material, the axial and bending stiffness of this beam will vary from point to point!.

In light of this trivial observation, we might want to reserve the term 'homogeneous' for, say, a circular beam made of an orthotropic material but with the atoms arranged regularly in a polar coordinate system as shown in Fig. 2. The stiffness properties of this beam are now clearly independent of position along the axis. But cut a small cube (still made of millions of atoms) and give it to a material scientist. This piece of material contains distributed dislocations (arrived at, perhaps, through a process of plasticity). We thus obtain a 'homogeneous' beam that is made of an inhomogeneous material. To avoid any ambiguities, we shall henceforth say that a beam enjoys the property of 'embedded homogeneity' if it is made of an actually homogeneous material, that is, if it can be conceived as having been carved out of a homogeneous material block. The problem of interest in this article consists of establishing necessary criteria for an elastic beam structure to enjoy the property of embedded homogeneity.

3. Statement of the problem

If a thin curved strip is cut from a homogeneous, possibly anisotropic, simple elastic plane sheet of stress-free material, the piece extracted is a stress-free plane elastic beam. Given its provenance, we would like to affirm that this beam is, in some sense, also homogeneous, just as the parent sheet. Let, however, the constitutive law of the sheet be given in terms of an elastic energy ψ per unit area

$$\psi = \psi(\mathbf{C}). \tag{1}$$

In this equation, $\mathbf{C}$ represents the right Cauchy-Green tensor, whose components in a fixed referential Cartesian coordinate system (X^1, X^2) we denote by

$$[\mathbf{C}] = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \tag{2}$$

Notice in Eq. (1) the conspicuous absence of an explicit dependence on position, in a manner consistent with the assumed homogeneity of the material in the stress-free reference configuration. With some abuse of notation, we may replace the constitutive law (1) by

$$\psi = \psi(a, b, c), \tag{3}$$

with the tacit understanding that we are working in a fixed coordinate system.

From the point of view of a theory of beams, such as the Bernoulli model that we assume adopted from now on, the constitutive equation of the beam can be obtained by restricting the possible deformations to those that keep the originally normal cross sections always undistorted and perpendicular to the current configuration of the beam axis. Let s and z be referential coordinates

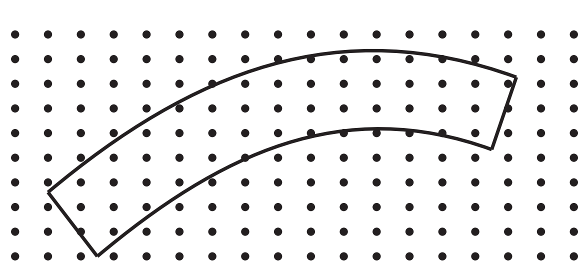

Fig. 1. A beam carved out of a homogeneous block.

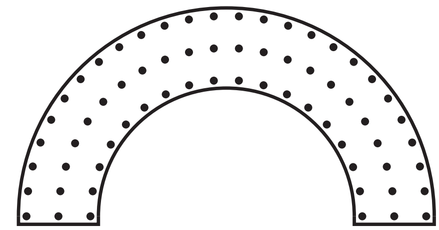

Fig. 2. A beam of constant stiffness.

measuring, respectively, length along the beam axis and position measured on the normal. In this orthogonal curvilinear coordinate system the element of area is expressed as

$$dA = (1 + \kappa z)\, ds\, dz, \tag{4}$$

where $\kappa = \kappa(s)$ is the curvature of the line $z = 0$, namely, the referential beam axis. Having chosen positive unit directions $\mathbf{t}$ for s and $\mathbf{n}$ for z, the (signed) curvature is obtained as

$$\kappa = \frac{d\mathbf{n}}{ds} \cdot \mathbf{t}. \tag{5}$$

Let the slope-angle at the material point s along the beam axis be denoted by $\theta = \theta(s)$. The right Cauchy-Green tensor field throughout the beam is given by

$$\mathbf{C}(s, z) = \mathbf{R}\Lambda\mathbf{R}^T, \tag{6}$$

where, in terms of components in the referential coordinate system, we have

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{7}$$

and

$$\Lambda = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}. \tag{8}$$

The entry $\lambda = \lambda(s, z) > 0$ in this matrix represents the axial measure of deformation, namely, the square of the stretch ratio. The dependence of λ on z is given by the formula

$$\lambda(s, z) = \lambda_0(s)\left(1 + \frac{\gamma z}{1 + \kappa z}\right)^2 = \lambda_0(s) H(\kappa, \gamma, z), \tag{9}$$

where $\gamma = \gamma(s)$ is the curvature change due to the deformation, and where $\lambda_0(s) = \lambda(s, 0)$. Combining all the above results, we obtain the elastic energy per unit referential length of the axis as

$$\phi(\lambda_0, \gamma, s) = \int_{-h/2}^{h/2} \psi\left(\begin{bmatrix} \lambda_0 H \cos^2\theta + \sin^2\theta & (\lambda_0 H - 1)\cos\theta\sin\theta \\ (\lambda_0 H - 1)\cos\theta\sin\theta & \lambda_0 H \sin^2\theta + \cos^2\theta \end{bmatrix}\right)(1 + \kappa z)\, dz, \tag{10}$$

where h is the (constant) thickness of the beam, assumed to be bisected by the beam axis.

In short, by a process of integration of the constitutive law of the original material of the sheet, it is possible to derive rigorously the elastic constitutive equation of the beam, in terms of the axial stretch ratio and the curvature change, as some function

$$\phi = \phi(\lambda_0, \gamma, s). \tag{11}$$

Notice that, in general, the resulting expression will contain an explicit dependence on the coordinate s. This dependence originates from the anisotropy of the material, as captured in the known slope angle $\theta = \theta(s)$, as well as from the known referential curvature $\kappa(s)$ of the beam.

Assume, instead, that the constitutive equation of a beam is given as the point of departure, namely, a smooth energy density