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Dispersion phenomena in symmetric pre-stressed layered elastic structures

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ABSTRACT

The dispersion relation associated with a symmetric three layer structure, composed of compressible, pre-stressed elastic layers, is derived. This mathematically elaborate transcendental equation gives phase speed as an implicit function of wave number. Numerical solutions are established to show a wide range of dispersion behaviour which is delicately dependent on the material parameters and pre-stress in each layer. Particularly interesting behaviour is observed within the short wave (high wave number) regime, with six possible cases of short wave liming behaviour shown possible. Within each of these, a short wave asymptotic analysis is carried out, resulting in a set of approximations which provide explicit relationships between phase speed and wave number. It is envisaged that these approximations may prove helpful to approximate numerical truncation errors associated with impact response, as well as providing excellent first approximations for particularly (numerically) challenging sets of material parameters.

1. Introduction

Many modern technological applications exist in which layered elastic material components are employed. Amongst the most common, we cite the use of layered composites within the aerospace industry and rubber-like vibration control devices used for earthquake protection in bridges and tall buildings. There are also examples of naturally occurring layered structures in geo and biomechanics. With these and other applications in mind, a significant research effort has in recent times been focussed on elucidation of the dynamic properties of layered elastic media. As the complexity of application has increased, considerable effort has been put into a complete understanding of dispersion in layered media. Seemingly the first attempt to consider non-fundamental modes was made in Lamb (1917) for a plane section of an isotropic plate. A more complete study of higher modes of an isotropic plate was later presented in Mindlin (1960). Because of the very complicated structure of the underlying dispersion relations, most of the results in these early papers were obtained through use of numerical computations.

In this present contribution, we aim to add to current understanding by investigating wave propagation in a symmetric 3-layer laminate, with each layer composed of compressible, pre-stressed elastic material. The paper will specifically focus on symmetric (extensional-type) waves and as such generalise the constitutive framework of a study done some years ago for incompressible pre-stressed elastic layers, see Rogerson and Sandiford (2000). Within this present paper a thorough investigation of the dispersion relation associated with the aforementioned structure is carried out. An initial numerical investigation is used to enable short wave asymptotic approximations, giving phase speed explicitly in terms of wave number, to be established. We remark that a detailed understanding of the behaviour of the dispersion relation is a critical pre-requisite in determining dynamic response. Although traditionally much effort has been afforded to fundamental modes, there is very good motivation to not neglect the harmonics. For example, the solution of any impact problem will generally involve contributions from each mode over the entire wave number regime. Moreover, important features, such as surface or interfacial waves, may arise from the cumulative affects of the harmonics, rather than result from the short wave limit of a single mode, see Rogerson (1992). We also remark that Kaplunov et al. (1998) and Kaplunov and Markushevich (1993) show that for certain types of vibration the contributions of higher modes is significant.

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The problem of dispersion in layered pre-stressed media has seen a number of publications over the last twenty years. Plane incremental waves in incompressible and compressible elastic single-layer plates (with traction free boundaries) were discussed in Ogden and Roxburgh (1993) and Roxburgh and Ogden (1994), respectively. An asymptotic short wave analysis for a single layer plate was carried out in Rogerson (1997), Sandiford and Rogerson (2000) and Nolde et al. (2004) for incompressible, nearly incompressible and compressible materials, respectively. An extension to three dimensional motion was carried in Pichugin and Rogerson (2002) in respect of incompressible elastic single layer plates. Many of the aforementioned papers are a good source of references to earlier work.

The paper is organised as follows. In Section 2 a very brief review of the underlying constitutive theory, and equations of motion, is presented. The dispersion relation is derived in Section 3. A numerical analysis of the dispersion relation is presented in Section 4. From this numerical analysis it is shown that the short wave limiting behaviour may be classified into six distinct cases. Each of these cases are analysed in Section 5, within which short wave asymptotic approximations are established in each case. These approximations, give phase speed as an explicit function of wave number and material parameters for each mode and are shown to provide excellent agreement with the numerical solution over a surprisingly large wave number regime. It is envisaged that these might provide some help in the numerical inversion of the highly oscillatory wave number integrals associated with impact problems.

2. Governing equations

Only a brief summary of the relevant underlying theory is presented; for further details the reader is referred to [8]. We begin by considering a homogeneous elastic body $\mathscr{B}$, possessing a natural isotropic unstressed state $\mathscr{B}_0$. A purely homogeneous static deformation is imposed, resulting in the pre-stressed equilibrium state $\mathscr{B}_e$. Upon $\mathscr{B}_e$, we superimpose a small time dependent motion $u_i(X_A,t)$, resulting in the current material state $\mathscr{B}_t$. Position vectors of a representative particle are denoted by X_A, $x_i(X_A)$ and $\tilde{x}_i(X_A,t)$ in $\mathscr{B}_0$, $\mathscr{B}_e$ and $\mathscr{B}_t$ respectively, with $\tilde{x}_i$ expressible in the form

$$\tilde{x}_i(X_A,t) = x_i(X_A) + u_i(X_A,t). \tag{1}$$

The deformation gradients associated with the deformations $\mathscr{B}_0 \to \mathscr{B}_t$ and $\mathscr{B}_0 \to \mathscr{B}_e$ are denoted by $\mathbf{F}$ and $\bar{\mathbf{F}}$ and take the component forms

$$F_{iA} = \frac{\partial \tilde{x}_i}{\partial X_A}, \quad \bar{F}_{iA} = \frac{\partial x_i}{\partial X_A}. \tag{2}$$

On making use of Eqs. (1) and (2), $\mathbf{F}$ and $\bar{\mathbf{F}}$ may be related through

$$F_{iA} = (\delta_{ij} + u_{i,j})\bar{F}_{jA}. \tag{3}$$

The most general isotropic strain-energy function is of the form

$$W(F) = W(I_1, I_2, I_3) \tag{4}$$

with I_1, I_2 and I_3 principal invariants of the left Cauchy–Green strain tensor. The equations of infinitesimal incremental motion may be written as

$$\pi_{iA,A} = \rho_0 \ddot{u}_i, \quad \pi_{iA} = \frac{\partial W}{\partial F_{iA}} \tag{5}$$

with ρ_0 the density per unit volume of $\mathscr{B}_0$, π_{iA} components of the first Piola Kirchhoff stress tensor and a superimposed dot indicating differentiation with respect to time. A linearised form (5) is obtainable in the form

$$\mathscr{A}_{piba} u_{a,bp} = \rho_e \ddot{u}_i \tag{6}$$

with ρ_e the material density per unit volume of $\mathscr{B}_e$ and $\mathscr{A}_{piba}$ the fourth order elasticity tensor, defined in the component form

$$\mathscr{A}_{piba} = J^{-1}\bar{F}_{bC}\bar{F}_{pA}\frac{\partial^2 W}{\partial F_{iA}\partial F_{aC}}\bigg|_{\mathbf{F}=\bar{\mathbf{F}}}. \tag{7}$$

The components of $\mathscr{A}$ allow especially simple representation relative to the principal axes of the primary static deformation. The only non-zero components have the form $\mathscr{A}_{iijj}$, $\mathscr{A}_{ijij}$ or $\mathscr{A}_{ijji}$ with $i,j \in \{1,2,3\}$. These are given in terms of the principal stretches λ_m, with $m \in \{1,2,3\}$, as

$$\mathscr{A}_{iijj} = J^{-1}\lambda_i\lambda_j\frac{\partial^2 W}{\partial\lambda_i\partial\lambda_j}, \tag{8}$$

$$\mathscr{A}_{ijij} = \begin{cases} J^{-1}\left(\lambda_i\frac{\partial W}{\partial\lambda_i} - \lambda_j\frac{\partial W}{\partial\lambda_j}\right)\frac{\lambda_i^2}{\left(\lambda_i^2-\lambda_j^2\right)} & i \neq j \;\; \lambda_i \neq \lambda_j, \\ \frac{1}{2}\left(\mathscr{A}_{iiii} - \mathscr{A}_{iijj} + J^{-1}\lambda_i\frac{\partial W}{\partial\lambda_i}\right) & i \neq j \;\; \lambda_i \neq \lambda_j, \end{cases} \tag{9}$$

$$\mathscr{A}_{ijji} = \mathscr{A}_{jiij} = \mathscr{A}_{ijij} - J^{-1}\lambda_i\frac{\partial W_0}{\partial\lambda_i} \quad i \neq j. \tag{10}$$

We also need a linearised measure of incremental traction, obtainable as

$$\tau_i = \mathscr{A}_{jilk}u_{k,l}n_j \tag{11}$$

with $\mathbf{n}$ the outward unit normal to a material surface in $\mathscr{B}_e$.

Our aim is to consider wave propagation in layered media, layers having finite thickness in one spatial direction and infinite lateral extent in the other two. Each layer is composed of elastic material characterised by the strain energy function (4). A Cartesian coordinate system $Ox_1x_2x_3$ is chosen, coincident with the principal axes of deformation in $\mathscr{B}_e$, with Ox_2 normal to the upper traction free surface. A state of plane strain is assumed, with both u_1 and u_2 independent of x_3 and $u_3 \equiv 0$. On making use of (8)–(10) in (6), the two non-trivial equations of motion are expressed as

$$\mathscr{A}_{1111}u_{1,11} + (\mathscr{A}_{1122} + \mathscr{A}_{2112})u_{2,21} + \mathscr{A}_{2121}u_{1,22} = \rho_e\ddot{u}_1, \tag{12}$$

$$(\mathscr{A}_{1221} + \mathscr{A}_{2211})u_{1,12} + \mathscr{A}_{1212}u_{2,11} + \mathscr{A}_{2222}u_{2,22} = \rho_e\ddot{u}_2. \tag{13}$$

We now seek solutions of (12) and (13) in the form

$$(u_1, u_2) = (U, V)e^{kqx_2}e^{ik(x_1 - vt)} \tag{14}$$

yielding two linear homogeneous equations with non-trivial solutions if

$$\alpha_{22}\gamma_2 q^4 + \{\alpha_{22}(\bar{v}^2 - \alpha_{11}) + \gamma_2(\bar{v}^2 - \gamma_1) + \beta^2\}q^2 + (\bar{v}^2 - \alpha_{11})(\bar{v}^2 - \gamma_1) = 0 \tag{15}$$

with $\bar{v}^2 = \rho_e v^2$ and α_{ij}, γ_i and β defined by

$$\alpha_{ij} = \mathscr{A}_{iijj}, \; i,j \in \{1,2\}, \quad \gamma_1 = \mathscr{A}_{1212}, \quad \gamma_2 = \mathscr{A}_{2121}, \quad \beta = \alpha_{12} + \gamma_2 - \sigma_2 \tag{16}$$

and with $\sigma_2 = \gamma_2 - \mathscr{A}_{1221}$ the principal Cauchy stress along Ox_2. If the two roots of equation (15) are denoted by q_1^2 and q_2^2, we note these may be either real, purely imaginary or complex conjugates, U and V may be presented as linear combinations of four generally independent solutions. On making use of equations (12) and (13), U and V may be expressed as

$$U = \sum_{m=1}^{2} U^{(2m-1)}e^{kq_m x_2} + U^{(2m)}e^{-kq_m x_2}, \tag{17}$$

$$V = \sum_{m=1}^{2} \frac{g(q_m)}{\beta} q_m \left(U^{(2m-1)}e^{kq_m x_2} - U^{(2m)}e^{-kq_m x_2 d}\right), \tag{18}$$