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Dynamic stability of spinning viscoelastic cylinders at finite deformation

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ABSTRACT

The study of spinning axisymmetric cylinders undergoing finite deformation is a classic problem in several industrial settings – the tire industry in particular. We present a stability analysis of spinning elastic and viscoelastic cylinders using ARPACK to compute eigenvalues and eigenfunctions of finite element discretizations of the linearized evolution operator. We show that the eigenmodes correspond to N-peak standing or traveling waves for the linearized problem with an additional index describing the number of oscillations in the radial direction. We find a second hierarchy of bifurcations to standing waves where these eigenvalues cross zero, and confirm numerically the existence of finite-amplitude standing waves for the nonlinear problem on one of the new branches. In the viscoelastic case, this analysis permits us to study the validity of two popular models of finite viscoelasticity. We show that a commonly used finite deformation linear convolution model results in non-physical energy growth and finite-time blow-up when the system is perturbed in a linearly unstable direction and followed nonlinearly in time. On the other hand, Sidoroff-style viscoelastic models are seen to be linearly and nonlinearly stable, as is physically required.

1. Introduction

The behavior of steady spinning bodies at finite deformation is of both theoretical and practical interest. In the special case that the body is axisymmetric, there has been a fair amount of work devoted to its formulation, elucidation, and behavior. Notable early work on the finite deformation spinning axisymmetric body is due to Padovan and Paramodilok (1983, 1985), Oden and Lin (1986), Padovan (1987), Bass (1987) and Kennedy and Padovan (1987). This literature formulates the equations of motion in the frame of reference of a non-spinning observer translating alongside the rotating body, and then to various degrees examines the equilibrium solutions of the motion as the spin rate is varied. The issues of contact with "roadways" and viscoelastic as well as elastic response are considered. The work of LeTallec and Rahier (1994), which followed this early work, provides the first clear description of the problem's kinematics and, by virtue thereof, lays the groundwork for the proper understanding of the issues associated with correctly specifying the constitutive response of a spinning body in a non-spinning frame of reference; see Govindjee and Mihalic (1998) for a discussion of this point, and the work of Faria et al. (1992), which shows that the issue was somewhat understood prior to these latter two works.

A special feature of the response of a steady spinning elastic cylinder is the existence of non-axisymmetric stationary solutions (standing waves) that appear as bifurcations from an axisymmetric branch in the configuration space of the body (Oden and Lin, 1986; Chatterjee et al., 1999). The determination of these bifurcation speeds can be performed by searching for the spin rates at which the tangent operator of the equations of motion becomes singular. Similar non-axisymmetric steady spinning solutions have also been reported upon in the case of finite deformation viscoelasticity (Padovan and Paramodilok, 1983, 1985; Kennedy and Padovan, 1987; Chatterjee et al., 1999). However, in the viscoelastic case, external forcing through contact with a roadway or counter-rotating cylinder is required for these states to remain steady in time (Chatterjee et al., 1999).

In the present work, we study the linearized dynamics about the axisymmetric state of a freely spinning elastic or viscoelastic cylinder (not in contact with a roadway) and interpret all the eigenvalues and eigenfunctions as giving information on the dynamic behavior of the system on perturbation. We study the effect of rotating the eigenfunctions about the origin and classify them according to their rotational symmetry group and the number of radial oscillations in a tensor product representation. We are not aware that this structure of the eigenfunctions has previ-

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ously been recognized. We also show that a typical eigenmode can be interpreted as a traveling wave that progresses around the cylinder in the lab frame with an angular velocity that depends on the imaginary part of the eigenvalue. In the elastic case, the real part of resolved eigenvalues (computed using ARPACK on a finite element discretization of the linearized evolution operator) is always found to be zero and the amplitude of the traveling wave remains constant in time. In the viscoelastic case, the wave decays (or grows) at a rate determined by the real part of the eigenvalue. In the special case of a zero eigenvalue, the traveling wave is actually stationary in time, indicating a potential bifurcation to a branch of finite-amplitude standing wave solutions. This is consistent with previous studies that predict such bifurcations when the tangent operator becomes singular.

The traveling wave description of the eigenmodes of the linearized problem provides a physical interpretation for the critical rotation speed ω_c beyond which bifurcations to standing wave solutions are possible. In the same way that a traveling water wave in a canal appears stationary to an observer riding alongside it on horseback (Russel, 1845), a traveling wave in a rotating elastic cylinder appears as a stationary solution in the lab frame if it travels backward through the medium at the speed of rotation. Below the critical speed, all modes of the linearized operator travel through the physical medium faster than the rotation rate, so a stationary solution is not possible. However, for special values of ω greater than ω_c, namely the values of ω where an eigenvalue of the tangent operator is zero, there is one wave traveling backward through the medium at the same rate that the medium travels forward. The fact that the bifurcation speeds ω accumulate at ω_c suggests that the traveling speed of a mode through the medium approaches a limiting value as the azimuthal wave number of the mode approaches infinity. In their monograph, Rabier and Oden (1989) offer a similar explanation in the incompressible case, crediting Faria with the insight, but using a half-plane analysis to predict wave speeds around the cylinder rather than interpreting eigenmodes as traveling waves.

In our framework, the traveling speeds of the eigenmodes are measured in the lab frame; thus, many highly complex spatial modes are found to travel slowly when $\omega > \omega_c$, since their speed through the medium is comparable to the speed of the medium in the opposite direction. This leads to an unusual and difficult eigenvalue problem in which the ordering of the eigenvalues along the imaginary axis has little correspondence with the spatial complexity of the modes. In light of these difficulties, it is remarkable that Rabier and Oden were able to prove existence of finite-amplitude standing wave branches in the incompressible case, with bifurcation points converging to ω_c from the right, using Lyapunov–Schmidt theory and Fredholm index theory.

Treating non-zero eigenvalues of the tangent operator on equal footing with the zero eigenvalues leads to many new questions. In particular, it is likely that families of finite-amplitude traveling solutions for the nonlinear problem bifurcate from non-zero (imaginary) eigenvalues. Computing such solutions would entail formulating the problem in a reference frame that rotates at the speed of the traveling wave in the lab frame, which is different than the rotation rate of the cylinder. Cyclic dynamics of the nonlinear problem may also result when bifurcations from superpositions of two linearized traveling waves exist. As a first step to exploring these possibilities, we show that one of the stationary modes that lies outside of the Oden and Lin hierarchy (due to a more complicated radial dependence) leads to a bifurcation branch of finite-amplitude solutions.

In the viscoelastic case, the eigenvalues have a non-zero real part, so rather than indicating bifurcations to pure stationary or traveling solutions, the eigenmodes also decay (or grow) in time at a rate determined by the magnitude (and sign) of the real part. The imaginary parts of these eigenvalues behave much the same as in the elastic case, with a critical frequency ω_c beyond which many modes emerge with complex spatial structure that travel slowly or remain stationary in the lab frame; however, due to the non-zero real part of the eigenvalue, these modes decay (or grow) as they travel. Exponential growth of well-resolved eigenmodes indicates that the system is linearly unstable. When this occurs, we explore nonlinear stability by seeding the stationary solution with a perturbation in the unstable direction and evolving the system through time according to the full nonlinear evolution equations. Finite-time blow-up indicates a deficiency in the underlying (viscoelastic) material model.

An important issue when considering the mechanical response of continuum bodies is that the constitutive relations that are selected must satisfy the Clausius–Duhem inequality expressing the second law of thermodynamics; see e.g. Coleman and Noll (1963) or Truesdell and Noll (1965, Section 79). In the finite deformation viscoelastic setting there are two approaches to setting up such constitutive relations. One, due to Coleman (1964a,b), is to construct a free energy functional of the history of the material whose derivative with respect to the current deformation gradient yields a history functional giving the stress response. This framework seems natural for convolution type viscoelastic models, such as the well-known BKZ model (Bernstein et al., 1963) and the Simo model (Simo, 1987). Notwithstanding the popular status of these two models, and the formal appearance of convolution expressions in their specification, the requisite free-energy functional that generates them has never been reported. In other words, while these models have the appearance of being strictly dissipative, it is actually not known if they satisfy the Clausius–Duhem inequality except in the infinitesimal strain limit, where they do. An alternate viscoelastic framework is provided by the work of Sidoroff (1974), who proposes a multiplicative split of the deformation gradient, similar to finite deformation plasticity models, and then directly constructs evolution laws for the viscoelastic variables that satisfy the Clausius–Duhem inequality. Well known examples of models of this type are due to LeTallec and Rahier (1994) and Reese and Govindjee (1998b).

The spinning body problem provides an ideal setting for a deep comparison of these two distinctly different modeling frameworks. In particular we are able to demonstrate that models of the Simo-class become unstable (both linearly and nonlinearly) at high rotation rates, leading to non-physical results. By contrast, models of the Sidoroff-class behave well in similar situations. This point is particularly relevant for analysis schemes that rely upon a steady spinning state of a system followed by transient computation – e.g. in the modeling of a tire traveling at high speed that encounters a bump in the road.

An outline of the remainder of the paper is as follows: In Section 2 we review the strong and weak formulations of the elastic spinning body problem in both the steady and unsteady cases and discuss its Hamiltonian structure. In Section 3 we revisit the well-studied elastic bifurcation case to show that our formulation is consistent with previous work. We then go further to elucidate the structure of linearized solutions and identify a second hierarchy of bifurcations that have not been reported in numerical studies to our knowledge, but were mathematically foreshadowed in the monograph of Rabier and Oden (1989). We also compute a finite-amplitude standing wave on one of the new bifurcation branches. With this background, in Section 4 we present two viscoelastic models in a form suitable for the study of spinning bodies. This is followed in Section 5 by a stability analysis of the behavior of spinning viscoelastic cylinders and the strong influence of the choice of viscoelastic modeling framework. In Section 6 we consider a full nonlinear stability analysis and show that convolution-type viscoelastic models can lead to non-physical