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Three-dimensional elasticity based on quaternion-valued potentials

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ABSTRACT

One of the most fruitful and elegant approach (known as Kolosov–Muskhelishvili formulas) for plane isotropic elastic problems is to use two complex-valued holomorphic potentials. In this paper, the algebra of real quaternions is used in order to propose in three dimensions, an extension of the classical Muskhelishvili formulas. The starting point is the classical harmonic potential representation due to Papkovich and Neuber. Alike the classical complex formulation, two monogenic functions very similar to holomorphic functions in 2D and conserving many of interesting properties, are used in this contribution. The completeness of the potential formulation is demonstrated rigorously. Moreover, body forces, residual stress and thermal strain are taken into account as a left side term. The obtained monogenic representation is compact and a straightforward calculation shows that classical complex representation for plane problems is embedded in the presented extended formulas. Finally the classical uniqueness problem of the Papkovich–Neuber solutions is overcome for polynomial solutions by fixing explicitly linear dependencies.

1. Introduction

1.1. Applications of potential theory

Well known numerical methods such as Finite Element Method (FEM) or Boundary Element Method (BEM) enable to solve various complex mechanical problems including non-linear problems (plasticity or other non-linear behaviors, contact problems, large displacements etc.). Isotropic linear elasticity is nevertheless a frequent problem in mechanical engineering. Potential theory developed since the late 19th century is still widely used in linear elasticity in 2D and 3D. Barber (2003) presents an overview of the fundamental potential theory for elasticity related among others to Airy, Boussinesq, Green, Zerna, Galerkin, Papkovich and Neuber names. New potential formulations for instance developed by Kashtalyan and Rushchitsky (2009) deal with inhomogeneous media.

Many practical applications rely on potential theory. Stress Intensity Factors (SIF) in the framework of linear fracture mechanics have been intensively studied. For example Sneddon and Lowengrub (1969) or Kassir and Sih (1973, 1975) proposed various analytical solutions based on potential theory. Dual integral equations were intensively used for mixed boundary value problems that arise in potential theory adapted for crack problems. An overview of useful methods is given by Sneddon (1966). Fully analytical or semi-analytical solutions have also been established for various elastic problems using potential theory. For instance, Ying et al. (1996) applied potential theory for a pressure vessels and piping. Chau and Wei (2000) proposed a semi-analytical solution (relying on truncated expansions into series of the potentials) of a finite solid circular cylinder subjected to arbitrary surface load. More recently potential theory has been used for applied industrial investigations. In the field of rolling process for instance, coupled thermo-elastic inverse solutions that interpret (in real time) measurements of stress and temperature done under the surface of a cylindrical tool have been proposed in 2D by Weisz-Patrault et al. (2011, 2012, 2013) and in 3D by Weisz-Patrault et al. (2013, 2014). Thus, the contact between the product and the tool can be characterized during the process. Experimental tests that confirm the feasibility of such an approach have been performed by Weisz-Patrault et al. (2012) and Legrand et al. (2012, 2013). This kind of recent works contributes to renew the interest for potential theory because of their practical and technical content.

Furthermore, numerical methods can also be developed on the basis of potential theory. Hintermüller et al. (2009) proposed a 3D potential based numerical method for cracks and contact problems. Potential theory adapted for numerical methods are completely meshless and can be suitable for problems where very steep stress gradients are obtained avoiding mesh refinement and

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long computation times issues that arise with FEM for instance. Cruse et al. (1969) proposed such a numerical algorithm based on potentials and singular integral equations. Morales et al. (2013) proposed more recently a potential based numerical solution for 2D problems, and Morales et al. (2012) focuses on numerical uniqueness of the Boussinesq and Timpe solutions.

1.2. Motivations for extended Muskhelishvili formulas

For plane problems one of the most elegant and fruitful approach has been developed by Muskhelishvili (1953). Complex plane is used and holomorphic $\mathbb{C}$-valued potentials are derived from bi-harmonic Airy potential and Goursat theorem. A presentation of the theory and practical methods has been given by Lu (1995). The main advantages are related to the holomorphy of the involved potentials, indeed expansion into series, Cauchy formula and conformal mapping techniques are available as well as singular integral equation techniques studied by Muskhelishvili (1953). Usually, for three-dimensional problems $\mathbb{R}$-valued harmonic or bi-harmonic potentials are used, known as Galerkin vector potential and Papkovich–Neuber potentials initially introduced by Papkovich (1932) and re-discovered by Neuber (1934). These potential representations are complete, thus one can prove the existence of the potentials as studied by Mindlin (1936), Gurtin (1962), Stippes (1969), Cong and Steven (1979), Millar (1984), Hackl and Zastrow (1988). Complete general solutions are also studied in the fundamental works by Slobodyansky (1954, 1959) and Wang et al. (2008) among others.

On the basis of Papkovich–Neuber potentials, this paper aims at establishing a generalized Muskhelishvili formula in three dimensions. There is no direct extension of the complex plane in 3D. However, the four dimensional algebra of quaternions (Definition 1) is a convenient extension of the complex plane. Extensive work has been done in this field and a suitable extension in higher dimensions of holomorphic functions has been defined and studied intensively. For instance the book of Gürlebeck et al. (2007) gathers standard knowledge about the algebra of real quaternions. A class of functions, called monogenic (Definition 3), presents interesting similarities with holomorphic functions defined in the complex plane. Thus several advantages of the classical formulas of Muskhelishvili (1953) in 2D are transposed in 3D with the presented potential formulation. Indeed, monogenic power series expansions studied for instance by Malonek (1990), Bock and Gürlebeck (2010), Bock (2012) and Laurent series expansions (see e.g. van Lancker (1999), Bock (2012)) as well as the Cauchy formula (e.g. Brackx et al. (1982)) are still available. Conformal mapping technics are more limited than in 2D, but Möbius transformations are still available as detailed by Sudbery (1979).

A second motivation is the disadvantage of Papkovich–Neuber representation that arises if polynomial solutions of exact degree n are considered for the displacement field. Indeed, Bauch (1981) showed that if very classical spherical harmonics are used for the Papkovich–Neuber potentials then $8n+4$ polynomial solutions are generated, but the dimension of the subspace of polynomial solutions of degree n is only $6n+3$. Thus, many solutions obtained with Papkovich–Neuber representation are linear dependent which can cause numerical stability problems. But fixing these dependencies in explicit formulas is very difficult. However, Bock and Gürlebeck (2009b) already proposed a representation of displacement field by means of two monogenic functions which is similar to the representation demonstrated in this paper. Then Bock and Gürlebeck (2009a) demonstrated that $8n+8$ polynomial solutions are generated by considering spherical monogenics for the two monogenic functions. But $2n+5$ are linear dependent and explicit formulas have been given. Thus, monogenic representations present the significant advantage (compared with classical Papkovich–Neuber representation) of allowing explicit formulas of linear dependencies when spherical harmonics (or monogenics) are used for the potentials. Thus, numerical stability is expected to be much better for numerical applications.

In this paper, the existence of the two monogenic potentials is proven a priori by using only mathematical tools related to differentials calculus alike classical proofs of Airy potentials, Muskhelishvili formulas or Papkovich–Neuber representation. Thus completeness is demonstrated and an elegant and very compact representation of the displacement and stress fields is obtained. Moreover body forces, thermal strain and residual stress are taken into account in the potential representation. Finally in Section 6, polynomial solutions are constructed and it is shown how the redundancy of polynomial systems can be overcome.

Furthermore Piltner (1987, 1988, 1989) contributed significantly to potential theory by developing an alternative complete representations of 3D isotropic elasticity based on complex functions. Piltner (2001) provided an overview of complex methods. He was using six holomorphic functions depending on three complex variables, defined as complex-valued linear functions on $\mathbb{R}^3$. These representations cover under certain restrictions on the parameters the known representation formulas for the plane case and there are also results to restrict the number of complex variables to one. Without going too much into the details it should be mentioned that these representations are deeply related to each other. The linear functions used by Piltner can be found in Whittaker (1903) and in the book by Whittaker and Watson (1927) as a tool to describe spherical harmonics. In this way they are related also to the representation of Legendre polynomials and associate Legendre functions which are nowadays mainly used for this purpose (see for instance Sansone (1959)).

In this paper, a different framework is used (algebra of real quaternions instead of complex plane) regarding to the advantages listed in this section. It should be noted that another potential solution for 3D Neumann and Dirichlet problems (surface tractions or displacements imposed at the surface) for a general elastic body is described in the book of Bui (2006). The solution relies on the Kelvin–Somigliana or Kupradze–Bashelishvili tensors (equivalent to the Green tensor for elastostatic) introduced by Kupradze (1965). On this basis a simple or double layer potential vector and an integral equation has been solved analytically (in the form of an absolutely convergent series) by Pham (1967). In this paper the extended Muskhelishvili formulas are not derived from these potentials, because this method does not rely on harmonic analysis.

1.3. Geometrical restrictions

Complete representations for displacements require geometrical restrictions due to constructions. These restrictions are relatively weak and related to the boundary value problem that has to be solved. More serious is the problem of redundancy in the representation formulae because this avoids the uniqueness of the representations. Analyzing for instance the classical Papkovich–Neuber representation then it is known already for a long time that under certain additional assumptions only three of the four harmonic functions are needed. Sokolnikoff (1956) showed that one of the three harmonic functions in the vector potential can be omitted (set to be zero) if the domain is normal with respect to the corresponding direction. The scalar potential can be removed if for $\nu \neq \frac{1}{4}$ the domain is star-shaped. What is not so much discussed is the question whether additional assumptions are necessary if one of the four functions should only be expressed as a linear combination of the other three. A good survey on results about the uniqueness of the representations can be found also in Cong (1995).