International Journal of Solids and Structures 51 (2014) 3439–3448

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Confirming Inextensional Theory

V.R. Seereeram, K.A. Seffen *

Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, United Kingdom

ARTICLE INFO

Article history:
Received 28 March 2014
Received in revised form 7 May 2014
Available online 24 June 2014

Keywords:
Inextensional
Thin plates
Developable
Theory
Experiments
Finite element analysis

ABSTRACT

Thin, initially-flat plates can deform inextensionally and elastically during large out-of-plane deformations. This paper revisits an analytical method for describing the developable shapes of displaced plate, in order to quantify and validate its effectiveness. Results from practical experiments and finite element analysis are compared to theoretical predictions from well-known examples, and excellent correlations are obtained.

1. Introduction

Inextensional Theory was developed by Mansfield (1955) nearly 60 years ago for predicting the large-displacement, elastic shape of transversely-loaded, thin-walled elastic plates, typically used in aircraft structures. Shallow, or small, displacement methods are inadequate because they do not deal with the in-plane, or membrane, stresses concomitant to the build-up of moderate deflections—even those of the order of the thickness of the plate. Under larger deflections, Mansfield argues that, for simply-held plates, these membrane stresses and hence, strains are eventually curtailed because significant in-plane forces cannot be transmitted from the supporting boundary of the plate into the bulk of the structure. In the limit, he assumes that the membrane strains are zero, which permits a simplification of the deformed shape of plate in obeyance to Gauss's *Theorema Egregium*, namely, that it becomes a *developable* surface.

Mansfield renders the surface relative to the initial flat state as a general conical displacement field, and then using calculus of variations, the spatial distribution of corresponding conical *generators* is found by maximising the strain energy stored in plate under load, leading to a governing differential equation of generator layout. The variational nature of this formulation with its requirement of general, non-parallel generators was first proposed by Maxwell almost a century before (Niven, 1890) but without a generalised framework for its solution, as delivered by Mansfield. Some years later, Mansfield recognises that his theory is analogous to the earlier Tension Field Theory of Wagner for computing the shape of wrinkled regions in thin-walled terrestrial structures mainly under in-plane shear loads. Consequently, parallels between both types of problem and their performances emerge, for example, a higher theoretical stiffness is predicted because the displacement field is prescribed in both.

Mansfield provides insightful solutions for a few of his cases, including tip-loaded cantilevers and end-loaded strips as idealised models of aircraft wings, where he focusses mainly on calculating the load–deflection responses. In the case of a triangular plate, he also extracts a rudimentary picture of the generator layout using a strain lacquer painted onto the surface, which cracks in the direction of principal tensile strains after loading (Mansfield and Kleeman, 1955). Theoretical predictions of generators successfully compare when they are overlaid in this picture and, importantly, the experiment also confirms that the layout is fixed only by the loading type and planform geometry, and not by the loading magnitude, when displacements are greater than the thickness of plate. In other cases, the layout is not confirmed directly by experiment; instead Mansfield exploits the analogy with Tension Field Theory by devising wrinkled specimens of the same geometrical proportions and equivalent boundary conditions, where the highly visible outline of crests and troughs normal to wrinkles gives equivalent information on the expected layout of generators in his bending experiments: see Mansfield (1989).

The properties of generators may be gleaned instead from the displacement field of the plate, which is then transformed into generator data. For accuracy and completeness, it is necessary to

* Corresponding author. Tel.: +44 1223 764137.
E-mail address: kas14@cam.ac.uk (K.A. Seffen).

http://dx.doi.org/10.1016/j.ijsolstr.2014.06.001
0020-7683/

obtain a highly-resolved, three-dimensional map of the entire deformed surface, and this is now possible with recent advances in photographic technology that we describe later. Therefore, one aim of this study is to validate Inextensional Theory directly at a displacement level by comparing predictions of conical generators with those computed from the measured data: this quantitative comparison is absent from the literature but we are mindful that it achieves same visual objective as Mansfield's single experiment with lacquer. In doing so, we underline the value of Inextensional Theory but we also aim to widen its appeal to other researchers in view of a recent resurgence in problems featuring developable plates and shells, beyond Mansfield's cantilever plates. For example, the large deformation of confined shells provides some insight into the quantum world of dislocation movement in materials (Cerda and Mahadevan, 2005) and into the efficient storage of DNA ribbons within cells (Giomi and Mahadevan, 2010); macroscopically, the random crumpling of paper can be described by developable regions interconnected by sharp ridges (Amar and Pomeau, 1997), and the ordered wrinkling of a buckled cylinders is similar to the well-known developable Yoshimura pattern in foldable tubes (Seffen and Stott, 2014). New corrugated structures made from developable strips connected together are one type of "morphing" structure, which combine highly directional compliance and stiffness for achieving large changes in shape whilst preserving structural integrity (Seffen, 2012); simple shells "growing" out of plane under imposed, so-called inelastic strains, must buckle into a variety of developable mode shapes for growth to proceed efficiently (Seffen and Maurini, 2013).

In the Appendix at the end of paper, the main details of Inextensional Theory and pertinent examples are repeated from Mansfield (1989) for completeness. In the following section, the relevant kinematical assumptions and definitions are outlined ahead of processing the geometrical data from experiments in Section 3. Two types of experiment are carried out on the same theoretical examples from the Appendix. The first are physical experiments on a triangular plate loaded by a force applied to its tip. The deformed state is accurately recorded using a laser-scanning camera, and the process of obtaining generator information from the measured displacements is carefully described. The method is effective but the maximum displacements that can be wrought are limited for reasons described, although we far exceed Manfield's nominal limit of a single thickness. Therefore, our second "experimental" study is finite element analysis, which allows us to circumvent some of the practical issues faced before and to test the robustness of the assumptions of Inextensional Theory in earnest. Most notably, the induced deformation can be much larger and geometrically non-linear, and we can apply end-wise moments. We therefore consider swept plates with a broad free edge to which end moments can be applied, as well as the previous tip-loaded triangular plates. All results are compared in Section 4, before finishing with a discussion and conclusions.

2. Kinematics

Following Mansfield, Fig. 1 shows the planform of a thin, cantilever plate of general outline, rigidly built-in along one straight edge. The absent loading is applied normal to the plate, and the deformed surface is taken to be developable, where straight-line generators can be drawn through every point on the surface. By definition, there is no twist and curvature along a generator, only curving across and normal to it. Elemental slices of deformed plate are bounded by adjacent generators, which do not have to be parallel, so each slice deforms into a element lying on part of a hypothetical conical surface. The curvature of this element varies inversely with distance, η, along the generator, where the origin of coordinate is taken to be the conical vertex; because the curvature at this point is infinitely large, the vertex must lie outside the planform, as shown. The vertices of successive generators form a locus known as the generatrix, and their inclination is measured by the angle, α, with respect to some arbitrary datum line with ordinate, X. Crucially, all geometrical parameters are specified with respect to the original flat plane even though parts of the deformed plate may lie well above or below it: without this specification, the kinematics are simply unwieldy and tractable solutions for the layout of generators expressed via (α, X) are not forthcoming. As we shall show, this does not undermine the accuracy of results, even for relatively large displacements. This specification also dictates that the layout of generators remains fixed and *independent* of the loading magnitude, provided linear elasticity prevails. As a corollary, we only need to perform a single set of measurements on a deformed, thin plate without measuring the load: this reduces the number of tests to be performed as well as simplifying the practical set-up.

In these tests, we accurately measure the displaced shape of plate in Cartesian space and then compute the changes in plate curvature: in finite element analysis, these curvatures are directly available. In order to compare directly to solutions from Inextensional Theory, these curvatures are converted into generators using a Mohr's circle of twisting curvature versus ordinary curvature (Calladine, 1983). For every point on a developable surface, the Mohr's circle passes through its own origin, giving way to one non-zero principal curvature—the local conical curvature, κ_1. However, the asymptotic nature of Inextensional Theory suggests that membrane strains everywhere may be very small and not absolutely zero. Of course, these could be measured directly (although not easily) but we note from Gauss that only their particular spatial variation within the surface affects the developable assumption. In other words, if there is *Gaussian*, *i.e.* double curvature, then membrane strains are significantly present. Commensurately, we assume that the Mohr's circle has a small second principal curvature, κ_2, and we define a corresponding measure of the degree of membranal stretching—the stretching ratio, *SR*—such that

$$SR = \left|\frac{\kappa_2}{\kappa_1}\right| \quad (1)$$

which is calculated throughout the plate.

It turns out that this ratio is always a small number, with only moderate increases in value close to the edges of plate where the assumed conical shape does not comply with the requirements of the free-edge boundary condition. A boundary layer forms in practice to facilitate this requirement, where the original proponent, Basset (1890), estimates the width of layer to be "comparable with the thickness [of plate]"; much later, Mansfield (1989) carries out a formal calculation and shows its width is of the order of $\sqrt{t\kappa}$, where t is the thickness and κ the change in curvature of the plate. Since the ratio is very small elsewhere, it becomes appropriate to define an acceptable threshold below which membrane effects may be assumed to be negligible. We can examine the performance of points "lying" above or below this threshold, in order to appreciate the prevalence of inextensionality and, when we vary the threshold, we can observe how the character of this distribution fares. For example, when the threshold is lowered, more data points are classified as being extensional, and they grow in number, primarily, from the boundary of the plate inwards. However, extra points in the middle of the plate away from any edges also become classified as extensional even though conical displacements are clearly evident there when the shape of plate is inspected visually. The reason is because the threshold now equates with the level of noise inherent during numerical processing of the data. After trial and error, we set the threshold to be 0.02,