International Journal of Solids and Structures 51 (2014) 2084–2095

Contents lists available at ScienceDirect

International Journal of Solids and Structures

journal homepage: www.elsevier.com/locate/ijsolstr

Some basic contact problems in couple stress elasticity

Th. Zisis [a], P.A. Gourgiotis [b,*], K.P. Baxevanakis [c], H.G. Georgiadis [c]

[a] Department of Civil Engineering, University of Thessaly, Volos GR-38336, Greece
[b] Department of Mechanical and Structural Engineering, University of Trento, Trento I-38123, Italy
[c] Mechanics Division, National Technical University of Athens, Zographou GR-15773, Greece

ARTICLE INFO

Article history:
Received 22 November 2013
Received in revised form 28 January 2014
Available online 25 February 2014

Keywords:
Contact
Indentation
Microstructure
Micromechanics
Couple-stress elasticity
Singular integral equations

ABSTRACT

Indentation tests have long been a standard method for material characterization due to the fact that they provide an easy, inexpensive, non-destructive and objective method of evaluating basic properties from small volumes of materials. As the contact scales in such experiments reduce progressively (micro to nano-scales) the internal material lengths become important and their effect upon the macroscopic response cannot be ignored. In the present study, we derive general solutions for three basic two-dimensional (2D) plane-strain contact problems within the framework of the generalized continuum theory of couple-stress elasticity. This theory introduces characteristic material lengths in order to describe the pertinent scale effects that emerge from the underlying microstructure and has proved to be very effective for modeling microstructured materials. By using this theory, we initially study the problem of the indentation of a deformable elastic half-plane by a flat punch, then by a cylindrical indentor, and finally by a shallow wedge indentor. Our approach is based on singular integral equations which have resulted from a treatment of the mixed boundary value problems via integral transforms and generalized functions. The results show significant departure from the predictions of classical elasticity revealing that it is inadequate to analyze indentation problems in microstructured materials employing only classical contact mechanics.

1. Introduction

Indentation tests have long been a standard method for material characterization due to the fact that they provide an easy, inexpensive, non-destructive and objective method of evaluating basic properties from small volumes of materials and thin films. A range of three-dimensional head-shapes are used including the sphere (Brinell test), the circular cone (Rockwell C), the three-sided pyramid (Berkovich) and the four-sided pyramid (Vickers), as well as a range of essentially two-dimensional indentors such as the wedge and the cylindrical (Fischer-Cripps, 2004; Johnson, 1985).

Studies have shown that there is a strong size effect on hardness in polycrystalline, cellular and polymer materials especially when the indent size is in the sub-micrometer depth regime. For example, the measured indentation hardness of metals and ceramics increases by a factor of two as the width of the indent is decreased from 10 to 1 μm (Ma and Clarke, 1995; Poole et al., 1996; Stelmashenko et al., 1993). In addition, indentation of thin films showed an increase in the yield stress with decreasing film thickness (Huber et al., 2002). Fleck et al. (1994) suggested that the size effect on hardness is related to the high stress/strain gradients present in shallow indentations. This dependence on stress/strain gradients can also be concluded from dislocation theory (Fleck et al., 1994). Moreover, Ma and Clarke (1995) observed that the variation of hardness with indentation size is consistent with a strain gradient plasticity model. In general, hardening of materials is due to the combined presence of geometrically necessary dislocations associated with plastic strain gradients and statistically stored dislocations associated with plastic strains. However, although strain gradients are extensively used to interpret the size effects in plastic deformation, they are also important for materials that deform elastically when the representative length of the deformation field becomes comparable to the lengths of the material microstructure. In fact, Maranganti and Sharma (2007) showed that gradient effects play a significant role in complex materials with coarse-grain structure. Indeed, Chen et al. (1998) developed a continuum model for cellular materials and concluded that the continuum description of these materials obeys a gradient elasticity theory of the couple-stress type. In the latter study, the intrinsic material length was naturally identified with the cell size. Moreover, gradient theories were successfully utilized in the past to

---

* Corresponding author. Tel.: +39 0461 282594; fax: +39 0461 282599.
E-mail address: p.gourgiotis@unitn.it (P.A. Gourgiotis).

http://dx.doi.org/10.1016/j.ijsolstr.2014.02.016
0020-7683/

model materials with microstructure like foams (Lakes, 1983) and porous solids (Lakes, 1993). Fleck and Shu (1995) showed the significance of strain gradient effects in the buckling of elastic fibers in composite materials. Size effects were also observed experimentally in post-buckling behavior of thin films (Fang and Wickert, 1994).

On the other hand, in indentation experiments at very small indentation depths, plastic flow does not occur until the equivalent strain reaches a critical yield value. In addition, the displacement recovered during unloading is largely elastic and for this reason the elastic contact theory is generally used in order to determine the elastic modulus from a simple analysis of the indentation load–displacement data (Pharr et al., 1992). Under these circumstances, the observed response of the material may be interpreted only through elasticity considerations. However, the classical elasticity theory includes no internal length scales and therefore is unable to predict the experimentally observed size effects. In fact, as the contact scales reduce progressively (micro to nano-scales) the internal material lengths become important and their effect upon the macroscopic response cannot be ignored. For this reason, generalized continuum theories, such as the micropolar theory, the couple stress theory, and the more general strain-gradient theory, may be employed to interpret the microstructure-dependent size effects on the elastic properties of the material. These theories capture the effects of microstructure by enriching the classical continuum with additional material characteristic length scales, and, thus, extending the range of applicability of the 'continuum' concept in an effort to bridge the gap between classical continuum theories and atomic-lattice theories. A recent review of generalized continuum theories can be found in Maugin (2010).

One of the most effective generalized continuum theories has proved to be that of couple-stress elasticity, also known as Cosserat theory with constrained rotations (Mindlin and Tiersten, 1962; Koiter, 1964). This theory is the simplest gradient theory in which couple-stresses make their appearance. The couple stress theory may be viewed as a generalization of classical elasticity theory and departs from the classical theory in several significant respects. In particular, the modified strain-energy density and the resulting constitutive relations involve besides the usual infinitesimal strains, certain strain gradients known as the rotation gradients. Also, the generalized stress–strain relations for the isotropic case include, in addition to the conventional pair of elastic constants, two new elastic constants, one of which is expressible in terms of a material parameter ℓ that has dimension of [length]. The presence of this length parameter, in turn, implies that the modified theory encompasses the analytical possibility of size effects, which are absent in the classical theory. A recent study by Bigoni and Drugan (2007) provides an interesting account of the determination of the couple-stress moduli via homogenization of heterogeneous materials. In addition, Beveridge et al. (2013) performed experiments and numerical simulations in heterogeneous materials loaded in bending and directly related the characteristic material lengths in couple-stress elasticity with the intrinsic geometrical structure of the samples. Experiments with phonon dispersion curves indicate that for most metals, the characteristic internal length is of the order of the lattice parameter, about 0.25 nm (Zhang and Sharma, 2005a). However, other small-molecule materials have larger internal characteristic lengths. For example, for the semiconductor gallium arsenide (GaAs), Zhang and Sharma (2005b) estimated a characteristic length of about 0.82 nm, while Reid and Gooding (1992) estimated a microstructural length for graphite of the order of 3.3 nm.

Couple-stress elasticity had already in the 60's and 70's some successful application on stress concentration problems concerning holes and inclusions. In recent years, there is a renewed interest in couple-stress theory dealing with problems of microstructured materials. This is due to the inability of the classical theory to predict experimental observed size effects and also due to the increased demands for manufacturing devices at very small scales. For instance, a multitude of problems concerning fracture, plasticity, dislocations, and wave propagation have been analyzed within the framework of couple stress and related gradient theories. Recent applications include work by, among others, Vardoulakis and Sulem (1995), Huang et al. (1999), Lubarda and Markenscoff (2000), Fleck and Hutchinson (2001), Georgiadis and Velgaki (2003), Grammenoudis and Tsakmakis (2005), Grentzelou and Georgiadis (2005), Radi (2007), Gourgiotis and Georgiadis (2008), Piccolroaz et al. (2012).

Regarding size effects in contact problems, we note that through the years various models have emerged in the literature to quantify the observed size effect in indentation. Most of these models are phenomenological in nature based on gradient plasticity ideas or on discrete dislocation concepts (see e.g. Poole et al., 1996; Begley and Hutchinson, 1998; Nix and Gao, 1998; Shu and Fleck, 1998; Wei and Hutchinson, 2003; Danas et al., 2012). Another approach in the context of classical plasticity theories considers the effect of several micromechanical lengths upon the macroscopic indentation response by directly incorporating the microstructural characteristics of the indented half-space through purely geometrical considerations (see e.g. Chen et al., 2004; Stupkiewicz, 2007; Fleck and Zisis, 2010; Zisis and Fleck, 2010).

On the other hand purely elastic indentation of materials is hard to achieve in practice (Larsson et al., 1996). Nonetheless, elasticity can be of interest in particular cases. In fact, there are materials such as polymers that exhibit significant size effects also in the elastic regime (Han and Nikolov, 2007; Nikolov et al., 2007). In general, as was pointed out by Maranganti and Sharma (2007), materials with explicit microstructure such as cellular materials, composites, ceramics, glassy and semi-crystalline polymers can be fruitfully modeled by using gradient type elasticity theories.

In the present paper, we deal with three basic plane-strain contact problems in couple-stress elasticity. It is remarked that Muki and Sternberg (1965) were the first to study the effects of couple-stresses upon the flat-punch indentation response employing the elaborate method of dual integral equations. In our work, we extend their study by considering different types of indentors and utilizing a more direct approach based on singular integral equations. It is worth noting that this is the first analytical approach in the literature examining the response of various types of indentors in 2D contact problems within the context of gradient type elasticity theories. The paper is organized as follows: Initially, we summarize the fundamental equations of couple-stress elasticity under plane-strain conditions. Then, we formulate the three plane-strain contact problems concerning the indentation of an elastic half-space by (i) a flat punch, (ii) a cylindrical indentor, and (iii) a wedge indentor. To obtain full field solutions the method of singular integral equations is utilized. More specifically, the integral equations have resulted from a treatment of the mixed boundary value problems via Fourier transforms and generalized functions. The integral equations are then solved by employing analytical and numerical considerations. In the final part, the results for the three different indentation methods are presented and the influence of microstructure upon the solution is discussed in detail.

We note that our analysis may have some genuine practical application in qualitatively identifying the influence of length scale effects in real materials and possibly even quantifying the length scale parameter itself. The requirement to identify such effects, at least qualitatively, by simple procedures is of real practical importance (Lakes et al., 1985).