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When is a symmetric body-hinge structure isostatic?

Bernd Schulze [a,*], Simon D. Guest [b], Patrick W. Fowler [c]

[a] Department of Mathematics and Statistics, University of Lancaster, Lancaster LA1 4YF, UK
[b] Department of Engineering, University of Cambridge, Trumpington Street, Cambridge CB2 1PZ, UK
[c] Department of Chemistry, University of Sheffield, Sheffield S3 7HF, UK

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ABSTRACT

A symmetry-extended mobility rule is formulated for body-hinge frameworks and used to derive necessary symmetry conditions for isostatic (statically and kinematically indeterminate) frameworks. Constructions for symmetric body-hinge frameworks with an isostatic scalar count are reported, and symmetry counts are used to examine these structures for hidden, symmetry-detectable mechanisms. Frameworks of this type may serve as examples for exploration of a symmetry extension of the (now proven) 'molecular conjecture'.

1. Introduction

Isostatic systems are both kinematically and statically determinate, and so are fixed in configuration, and have no internal stresses when unloaded, thus allowing high precision placement of components, as discussed by Maxwell for scientific apparatus in Section 4 of Maxwell (1876). This has particular engineering relevance in harsh thermal environments such as space (Bujakas and Rybakova, 1998). Isostatic systems are able to react to changes in shape of their constituent bodies by deforming without building up internal stresses, and hence find application as 'parallel' robots, such as the Stewart platform (Stewart, 1965), deployable structures (Miura et al., 1985) and easily driven adaptive structures (Baker and Friswell, 2009).

In general, symmetry arguments give powerful tools for the detection of hidden mechanisms in structures that scalar counting arguments would predict to be isostatic. There are also many examples of highly symmetric structures that counting without symmetry predicts to be over-constrained, but which have mechanisms that are revealed by symmetry-extended counting rules (Röschel, 2002, 2012; Chen et al., 2012). The symmetry approach has already been used to develop symmetry-extended mobility criteria for bar-and-joint (Fowler and Guest, 2000; Connelly et al., 2009) and body-bar (Guest et al., 2010) frameworks. Here we make a natural extension to body-hinge structures, as a way of finding the symmetries of their mechanisms and states of self stress and identifying conditions for a symmetric structure of this type to be isostatic. Two simple examples of body-hinge structures are shown in Fig. 1.

In addition to the practical goal of explaining mechanisms in particular systems, the study of symmetry aspects of body-hinge structures has another motivation. The long-standing 'molecular conjecture' (Tay and Whiteley, 1984) was recently proved (Katoh and Tanigawa, 2011): under generic conditions, a body-hinge framework and a 'molecular' structure with the same underlying multi-graph have the same rigidity properties. (A *molecular structure*, named by analogy with chemical structures, is one in which the lines of the hinges attached to each body all pass through a common point in that body. Fig. 1(b) shows an example based on the propane molecule.)

A recently proposed generalisation is the conjecture that symmetric body-bar, body-hinge and molecular structures that all share a common symmetry and a common underlying multi-graph will have the same rigidity properties under symmetry-generic conditions (Porta et al., 2014). Comparison of the analogous symmetry counts for body-bar frameworks and the molecular structures that result from specialisation of the body-hinge systems considered in the present study could provide extra evidence for the 'symmetric molecular conjecture'. The present study also gives ways of quickly constructing examples of symmetric molecular structures with small numbers of mechanisms, for comparison with corresponding body-bar frameworks, hence furnishing examples for investigation of the symmetric molecular conjecture.

The plan of the paper is as follows. First, the symmetry-extended mobility rule for body-hinge structures is obtained for

* Corresponding author. Tel.: +44 (0) 1524 592173; fax: +44 (0) 1524 592681.
E-mail address: b.schulze@lancaster.ac.uk (B. Schulze).

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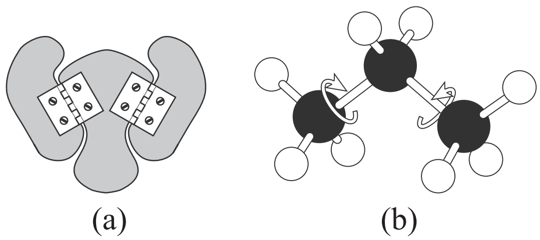

Fig. 1. Two example body-hinge structures: (a) shows a central panel connected to two outer panels through simple rotational hinges; (b) shows a model of a propane molecule in which each of the outer methyl groups is able to rotate about the bond to the central carbon atom. Each of these structures has two mechanisms, as in each case the hinges are able to rotate independently of one another.

assemblies of bodies pairwise connected by revolute hinges. Secondly, we derive general symmetry constraints on isostatic structures of body-hinge type. Finally, we present constructions for systems that are predicted to be isostatic by counting without symmetry, and examine their symmetry counts to determine whether they have symmetry-detectable mechanisms that are hidden by the scalar count.

In what follows, it is assumed that we are working with frameworks in three dimensions, except when specifically stated that we are dealing with the restriction of the system to the plane.

2. Background

The simple counting rule for calculating to first order the degrees of freedom (or the *mobility*) m of a mechanical linkage with b bodies connected by g joints, where joint i permits f_i degrees of freedom, is associated with Grübler and Kutzbach and was given in the following form by Hunt (1978):

$$m = 6(b-1) - 6g + \sum_{i=1}^{g} f_i. \tag{1}$$

The generalised version of this rule that allows for states of self-stress in the same way as Calladine's extension (Calladine, 1978) of Maxwell's Rule for bar-and-joint frameworks (Maxwell, 1864) is

$$m - s = 6(b-1) - 6g + \sum_{i=1}^{g} f_i, \tag{2}$$

where s is the dimension of the space of self-stresses of the linkage. Eq. 2 can be derived by considering the dimensions of the four fundamental vector subspaces of an equilibrium/compatability matrix, which can be defined for any set of linearised constraints (see, e.g., Guest and Pellegrino (1994) for an example).

A joint which allows exactly one revolute degree of freedom between the two bodies that it joins is called a *hinge*. Moreover, a mechanical linkage is called a *body-hinge structure* if every joint of the linkage is a hinge. Our goal is to derive necessary conditions for a symmetric body-hinge structure to be isostatic, i.e., to have $m = s = 0$.

Note that for a body-hinge structure with b bodies and h hinges, (2) becomes

$$m - s = 6(b-h-1) + h = 6b - 6 - 5h. \tag{3}$$

(In the restriction to two dimensions, the RHS is $3(b-h-1) + h = 3b - 3 - 2h$.)

3. A symmetry-extended mobility rule for body-hinge structures

The symmetry-extended version of the generalised mobility rule (2) is (Guest and Fowler, 2005):

$$\Gamma(m) - \Gamma(s) = (\Gamma_T + \Gamma_R) \times (\Gamma(v, C) - \Gamma_{\parallel}(e, C) - \Gamma_0) + \Gamma_{\mathrm{f}}, \tag{4}$$

where each Γ is the vector of the traces of the corresponding representation matrices in some point group $\mathcal{G}$. Each such Γ is known in applied group theory as a *representation* of $\mathcal{G}$ (Bishop, 1973), or in mathematical group theory as a *character* (James and Liebeck, 2001). In applied group theory, the term character is often used informally for denoting an entry of a representation, i.e., the trace of a representation matrix (Cotton, 1990) for a given operation.

In (4), $\Gamma(m)$ and $\Gamma(s)$ are the representations of the mobility and the states of self-stress, respectively. Γ_T and Γ_R are the representations of rigid-body translations and rotations, and can be read off from standard character tables for point-groups (Atkins et al., 1970; Altmann and Herzig, 1994). In 3D, $\Gamma_T + \Gamma_R$ is the six-dimensional $\Gamma(T_x, T_y, T_z) + \Gamma(R_x, R_y, R_z)$; in 2D, $\Gamma_T + \Gamma_R$ is the three-dimensional $\Gamma(T_x, T_y) + \Gamma(R_z)$, where the system lies in the xy plane. Γ_0 denotes the trivial representation which takes the value of one for all group elements.

The other representations are defined in terms of the so-called (Guest and Fowler, 2005) *contact polyhedron* C associated with the given body-hinge structure. C has one vertex for each body of the structure and two vertices are joined by an edge of C iff the corresponding bodies are connected by a joint. There is some choice in the construction of C, as we discuss further below. Note that C is not always a polyhedron in the graph theoretical sense: in some cases it may correspond to a planar graph, and in some to a non-planar graph. Further, its geometric embedding may have non-planar faces, or even degenerate to a polygon. $\Gamma(v, C)$ is the permutation representation of the vertices of C, and $\Gamma_{\parallel}(e, C)$ is the representation of a set of vectors along the edges of C. Finally, Γ_{f} is the representation of the total set of freedoms allowed by the joints.

Our previous treatments of mobility (Guest and Fowler, 2005) deals with hinges of all types, including sliders and screws, but in the present context, for a body-hinge structure, the symmetry-extended mobility rule (Guest and Fowler, 2005) equivalent to (3) is

$$\Gamma(m) - \Gamma(s) = (\Gamma_T + \Gamma_R) \times (\Gamma(v, C) - \Gamma_{\parallel}(e, C) - \Gamma_0) + \Gamma_{\mathrm{h}}, \tag{5}$$

where Γ_{h} is the representation of the revolute degrees of freedom allowed by the hinges (which we will determine below).

The form of the product on the RHS of (5) has one immediate consequence: as the multiplier $(\Gamma_T + \Gamma_R)$ has character zero under all improper operations (i.e., inversion, reflections or rotoreflections), the character of $\Gamma(m) - \Gamma(s)$ under such operations is determined entirely by that of the hinge freedoms for those operations. A second deduction can be made about frameworks that have an isostatic count of bars and hinges *and* have no body or hinge lying on an element of symmetry. Following the reasoning applied to bar-and-joint frameworks in Fowler et al. (in press), in the present case it is the bodies and hinges that fall into *orbits* of size $|\mathcal{G}|$, and all vertex, edge and hinge representations are multiples of Γ_{reg} (which has character $|\mathcal{G}|$ under the identity, and zero under all other symmetry operations). Thus,

$$\Gamma(v, C) = b_0 \Gamma_{\mathrm{reg}},$$
$$\Gamma_{\parallel}(e, C) = \Gamma_{\mathrm{h}} = h_0 \Gamma_{\mathrm{reg}},$$

with $b_0 = b/|\mathcal{G}|$ and $h_0 = h/|\mathcal{G}|$. Since the framework has an isostatic count under the identity operation, we have (in 3D)

$$(6b_0 - 5h_0)|\mathcal{G}| = 6. \tag{6}$$