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Symbolic and numerical solution of the axisymmetric indentation problem for a multilayered elastic coating

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ABSTRACT

This paper is concerned with a semi-analytical approach to the solution of the axisymmetric indentation problem for a multilayered elastic half-space. The stress and displacement fields for each layer and the substrate are derived in closed form by using the Papkovich–Neuber potentials and the Hankel transform. The bonded or sliding interface conditions between the sub-layers are handled by the use of the appropriate transfer matrix, and then the mixed boundary value problem is reduced to a Fredholm integral equation. Symbolic and numerical computations of the solution are implemented in the symbolic software Mathematica in the form of a fast and efficient numerical algorithm, allowing rapid determination of the load–displacement curves and composite elastic properties for an arbitrary rigid indenter shape. A series of results for different indenters (flat, conical, spherical and blunted conical punch shapes) and different multilayered composites is presented and discussed.

The complete set of symbolic and numerical computations are provided as supplementary resources with the paper.

1. Introduction

Thin films and surface coatings are of a great importance in the context of many engineering applications, e.g. for the improvement of resistance to wear, increasing the strength and toughness of structural surfaces and the protection of solids in high temperature or corrosive environments. Multilayered or continuously graded materials offer the potential for further progress in optimized design of surface coatings. For the purpose of design optimization, the knowledge of properties of such materials is crucial. Undoubtedly, indentation is the approach most widely used for the identification of thin film properties spanning the range of scales from nanometer to macroscopic. The evaluation procedure is based on the analysis of the indentation curve P–h representing the applied load P on the indenter with respect to its penetration depth h during the loading/unloading test. The simplest approach to the problem would be to find suitable laws that describe some parts of the indentation curve, and then to extract the material properties as fitting parameters. However, even in the presence of such analytical descriptions, the identification of material properties through indentation analysis is a difficult inverse problem. As in most cases of inverse analysis, the direct problem has to be addressed first.

The standard method used in the direct approach to evaluating Young's modulus of a homogeneous bulk substrate was initially developed by Oliver and Pharr (1992) and improved later in Oliver and Pharr (2004). They proposed a relationship between the initial unloading slope of the P–h curve and the substrate's Young's modulus. The remaining parameters were obtained by a Finite Elements Analysis (FEA). Using the Oliver and Pharr framework, Dao et al. (2001) found a complete set of explicit analytical functions using dimensional analysis and then FEA. Those functions help solving the direct problem, i.e. finding the parameters that describe the loading/unloading slopes of the indentation curve. Also, these functions can be used for the inverse analysis of the indentation test. These procedures have been extended for extracting materials properties of anisotropic solids (Delafargue and Ulm, 2004; Vlassak and Nix, 1994; Swadener and Pharr, 2001; Vlassak et al., 2003).

For heterogeneous materials, some analytical solutions can be found in literature. Giannakopoulos and Suresh (1997a,b) developed solutions for several indenter tips under the assumption that the depth distribution of the substrate's Young's modulus follows a power law or exponential law. Then Choi et al. (2008a,b) extended this approach to plastically graded materials using a similar approach to that of Dao et al. (2001). Nevertheless, often no a priori assumption can be made regarding the depth distribution of properties, and hence a multi-layer approach is necessary. Ke and Wang (2006) developed semi-analytical solutions for the plane strain problem using linear piecewise property distributions in each

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layer. However, the indentation problem is most often approximated as axisymmetric. For such problems some solutions have also been documented in the literature, e.g. the elastic solution of a homogenous layered half space with perfect interfacial bonding under axisymmetrical compressive loading carried out by Li and Chou (1997). Due to the complexity of the integrals involved in the use of the Hankel transform, the elastic fields for the coating/substrate system were obtained by a numerical inversion procedure. A different method of solution for the indentation of a thin film deposited on an elastic substrate and indented by an axisymmetric rigid punch was presented by Yu et al. (1990). This approach used the method proposed by Lebedev and Uflyand (1958) which results in the formulation of a Fredholm integral equation. The merit of this method of analysis lies in its ease of numerical implementation and the possibility of its application to a large range of different indenter tips. Fretigny and Chateauminois (2007) studied the problem with a constant piecewise distribution of elastic parameters by adopting a matrix formalism. They investigated the case of one layer over a substrate, but gave directions on how to generalize the method of solution to multilayered solids. Tang et al. (2008, 2009) studied experimentally and numerically the elastic modulus of metal-ceramic nanolaminates measured by axisymmetric nanoindentation. The elastic modulus of the multilayer was obtained according to the method proposed by Oliver and Pharr (1992). Recently, Korsunsky and Constantinescu (2009) used the technique of Yu et al. (1990) in order to study the influence of punch tip sharpness on the interpretation of indentation measurements for the layered elastic half-space. Perfectly bonded or freely sliding boundary conditions between the film and the substrate were taken into account. The authors considered axisymmetric indenters with different tip shapes, namely, the flat punch, spherical indenter, as well as conical and blunted conical indenters.

In this work, we consider the frictionless axisymmetric indentation of a multilayer lying on a semi-infinite elastic substrate. This problem can be seen as the indentation of an elastically graded material with constant piecewise distributions. Extending Korsunsky and Constantinescu's (2009) approach for a single layer to an arbitrary number of layers on a dissimilar substrate, we present a symbolic/numerical method of solution of the direct problem for several rigid indenter shapes. The use of symbolic computation permits several goals to be achieved, namely: (i) to verify the accuracy of the closed form solutions and to eliminate coding errors, (ii) to create a numerical code that performs the computation for n layers, where the number n is not predefined. Compared to the Finite Element Method, this approach is not limited by the size of elements that represent the thickness of the layers, and a better parametric understanding of the deformation phenomena during indentation becomes possible.

2. The multilayer coating indentation problem

Let us consider a multilayer composed of n layers deposited upon a half-space and being indented by an axisymmetric rigid punch as shown in Fig. 1. Both the layers and and the half-space are assumed to be homogeneous with a linear isotropic elastic behavior. The layers are indexed by $i = 1, n$ with increasing depth, each characterized by thickness h_i, shear modulus μ_i and Poisson's ratio ν_i.

For the rigid, axially symmetric indenter, four different shapes are considered next: the flat punch, the spherical cap, the sharp conical indenter and the blunted conical indenter (see sketches in Fig. 2).

The contact is considered to be frictionless and the problem is treated under the assumption of small strains. Cylindrical coordinates (r, θ, z) are used, with $z > 0$ pointing into the substrate, and each problem posed in terms of (i) the elasticity equations for each layer and the half space, (ii) the boundary conditions between layers, and (iii) the contact boundary conditions between the indenter and the first layer.

The elastic displacement and stress fields will be expressed in terms of the Papkovich–Neuber displacement potentials, given by the harmonic vector and the scalar function $\underline{\Psi}^j = \left(0, 0, \Psi^j\right)$ and ϕ^j respectively. (For a complete presentation of the Papkovich–Neuber potentials see, for example (Constantinescu and Korsunsky, 2007; Robert and W., 1999; Solomon, 1968)). The harmonicity of the potentials insures that the equations of linear isotropic elasticity are satisfied in each layer.

The elastic displacement and stress fields are given by:

$$2\mu_j u_r^j = -\phi^j_{,r} - z\psi^j_{,r} \tag{1}$$

$$2\mu_j u_z^j = k_j\psi^j - \phi^j_{,z} - z\psi^j_{,z} \tag{2}$$

$$\sigma^j_{zz} = 2(1-\nu_j)\psi^j_{,z} - \phi^j_{,zz} - z\psi^j_{,zz} \tag{3}$$

$$\sigma^j_{rz} = (1-2\nu_j)\psi^j_{,r} - \phi^j_{,rz} - z\psi^j_{,rz} \tag{4}$$

where $\kappa_j = 3 - 4\nu_j, u_r^j, u_z^j$ denote the components of the displacement vector, $\sigma^j_{zz}, \sigma^j_{rz}$ are the components of the stress tensor and the superscript j denotes the number of layer, and refer to the substrate if $j = n+1$.

Under the assumption of axial symmetry, the harmonic potentials ψ^i and ϕ^i can be expressed as the Hankel transform of four unknown arbitrary functions (Yu et al., 1990) $A_1^i(\lambda), A_2^i(\lambda), A_3^i(\lambda), A_4^i(\lambda)$.

The potentials in each layer i, for $(r,z) \in [0, +\infty[\times[z_{i-1}, z_i]$, are given by:

$$\psi^i(r,z) = \int_0^\infty \left(A_1^i \cosh(\lambda(z - z_{i-1}) + A_2^i \sinh(\lambda(z - z_{i-1}))\right) \times \frac{J_0(\lambda r)}{\sinh(\lambda(z_i - z_{i-1}))}\, d\lambda \tag{5}$$

$$\phi^i(r,z) = \int_0^\infty \left(A_4^i \cosh(\lambda(z - z_{i-1})) + A_3^i \sinh(\lambda(z - z_{i-1}))\right) \times \frac{J_0(\lambda r)}{\lambda \sinh(\lambda(z_i - z_{i-1}))}\, d\lambda \tag{6}$$

where sinh() and cosh() are the hyperbolic sine and cosine functions and J_0 is the Bessel function of first kind of order zero.

For the substrate, the expressions for the potentials are:

$$\psi^{n+1}(r,z) = \int_0^\infty A_5(\lambda)\exp(-\lambda(z - z_n))J_0(\lambda r)\, d\lambda \tag{7}$$

$$\varphi^{n+1}(r,z) = \int_0^\infty A_6(\lambda)\exp(-\lambda(z - z_n))\frac{J_0(\lambda r)}{\lambda}\, d\lambda \tag{8}$$

It may be seen that the expressions for the potentials in (6)–(8) ensure that stresses and their derivatives vanish in the layers for $z > 0$ if $r \to \infty$ and in the substrate for $r > 0$ if $z \to \infty$.

The boundary conditions between two successive layers i and $i+1$ are described either as a *perfect bonding* or as *frictionless sliding*.

Perfectly bonded layers impose the continuity of displacements and surface tractions (hence continuity of stress components zz and rz) at the respective interface:

$$u_z^i(r,h) = u_z^{i+1}(r,h) \tag{9}$$

$$u_r^i(r,h) = u_r^{i+1}(r,h) \tag{10}$$

$$\sigma^i_{zz}(r,h) = \sigma^{i+1}_{zz}(r,h) \tag{11}$$

$$\sigma^i_{rz}(r,h) = \sigma^{i+1}_{rz}(r,h) \tag{12}$$

Frictionless sliding between layers imposes the continuity of normal components of displacement and surface tractions and the vanishing of tangential surface traction: