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Strain energy change to the insertion of inclusions associated to a thermo-mechanical semi-coupled system

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ABSTRACT

The topological derivative measures the sensitivity of a given shape functional with respect to an infinitesimal singular domain perturbation. According to the literature, the topological derivative has been fully developed for a wide range of one single physical phenomenon modeled by partial differential equations. In addition, the topological asymptotic analysis associated to multi-physics problems has been reported in the literature only on the level of mathematical analysis of singularly perturbed geometrical domains. In this work, we present the topological derivative in its closed form for the total potential mechanical energy associated to a thermo-mechanical semi-coupled system, when a small circular inclusion is introduced at an arbitrary point of the domain. In particular, we consider the linear elasticity system (modeled by the Navier equation) coupled with the steady-state heat conduction problem (modeled by the Laplace equation). The mechanical coupling term comes out from the thermal stress induced by the temperature field. Since this term is non-local, we introduce a non-standard adjoint state, which allows to obtain a closed form for the topological derivative. Finally, we provide a full mathematical justification for the derived formulas and develop precise estimates for the remainders of the topological asymptotic expansion.

1. Introduction

The topological derivative represents a first order asymptotic correction term of a given shape functional with respect to a singular domain perturbation (Sokołowski and Żochowski, 1999). It has been applied in topology design optimization (Amstutz et al., 2012), inverse problems (Hintermüller et al., 2012), image processing (Hintermüller and Laurain, 2009), multi-scale constitutive modeling (Giusti et al., 2009), fracture mechanic sensitivity analysis (Van Goethem and Novotny, 2010) and damage evolution modeling (Allaire et al., 2011). See also the book by Novotny and Sokołowski (2013) and references therein.

For the sake of completeness, we recall the basic concepts on topological sensitivity analysis. Let us consider a bounded domain $\Omega \subset \mathbb{R}^2$, which is subject to a non-smooth perturbation confined in a small region $\omega_\varepsilon(\hat{x}) = \hat{x} + \varepsilon\omega$ of size ε, as shown in Fig. 1. Here, $\hat{x}$ is an arbitrary point of Ω and ω is a fixed bounded domain of $\mathbb{R}^2$. Associated to the domain Ω we introduce a characteristic function $x \mapsto \chi(x)$, $x \in \mathbb{R}^2$, namely $\chi = \mathbb{1}_\Omega$. Also, for the topologically perturbed domain we can define a characteristic function of the form $x \mapsto \chi_\varepsilon(\hat{x}; x)$. If the perturbation is given by a perforation, the characteristic function can be written as $\chi_\varepsilon(\hat{x}) = \mathbb{1}_\Omega - \mathbb{1}_{\overline{\omega_\varepsilon(\hat{x})}}$ and the perforated domain is obtained now as $\Omega_\varepsilon = \Omega \setminus \overline{\omega_\varepsilon}$. Now, by assuming the following topological asymptotic expansion of a given shape functional $\psi(\chi_\varepsilon(\hat{x}))$, associated to the topologically perturbed domain,

$$\psi(\chi_\varepsilon(\hat{x})) = \psi(\chi) + f(\varepsilon)D_T\psi(\hat{x}) + o(f(\varepsilon)), \tag{1}$$

the function $\hat{x} \mapsto D_T\psi(\hat{x})$ is called the topological derivative of ψ at $\hat{x}$. In (1), $\psi(\chi)$ is the shape functional associated to the original (unperturbed) domain and $f(\varepsilon)$ is a positive function such that $f(\varepsilon) \to 0$, when $\varepsilon \to 0$. After rearranging (1) we have

$$\frac{\psi(\chi_\varepsilon(\hat{x})) - \psi(\chi)}{f(\varepsilon)} = D_T\psi(\hat{x}) + \frac{o(f(\varepsilon))}{f(\varepsilon)}. \tag{2}$$

The limit passage $\varepsilon \to 0$ in the above expression leads to

$$D_T\psi(\hat{x}) = \lim_{\varepsilon \to 0} \frac{\psi(\chi_\varepsilon(\hat{x})) - \psi(\chi)}{f(\varepsilon)}. \tag{3}$$

Since we are dealing with singular domain perturbations, the shape functionals $\psi(\chi_\varepsilon(\hat{x}))$ and $\psi(\chi)$ are associated to topologically different domains. Therefore, the above limit is not trivial to be calculated. In particular, we need to perform an asymptotic analysis of the shape functional $\psi(\chi_\varepsilon(\hat{x}))$ with respect to the small parameter

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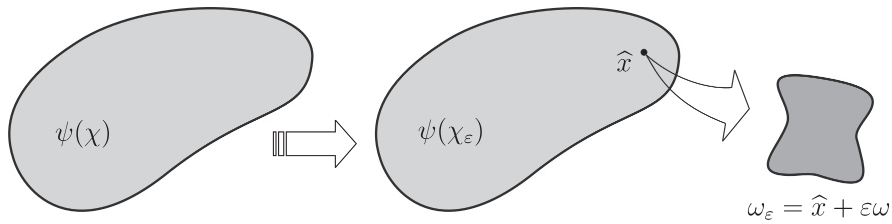

Fig. 1. The topological derivative concept.

ε. In order to calculate the topological derivative, in this work we will apply the approach fully developed in the book by Novotny and Sokołowski (2013). The method is based on the following result, whose rigorous mathematical justification can be found in the paper by Nazarov and Sokołowski (2003):

$$D_T\psi(\hat{x}) = \lim_{\varepsilon\to 0}\frac{1}{f'(\varepsilon)}\frac{d}{d\varepsilon}\psi(\chi_\varepsilon(\hat{x})). \tag{4}$$

The derivative of $\psi(\chi_\varepsilon(\hat{x}))$ with respect to ε can be seen as the sensitivity of $\psi(\chi_\varepsilon(\hat{x}))$, in the classical sense of Delfour and Zolésio (2001) and Sokołowski and Zolésio (1992), to the domain variation produced by an uniform expansion of the perturbation ω_ε.

According to the literature, the topological derivative has been fully developed for a wide range of one single physical phenomenon modeled by partial differential equations. In addition, only a few works dealing with multi-physics problems have been reported in the literature, and, in general, the derived formulas are presented in their abstract forms (see, for instance, the paper by Cardone et al. (2010) on topological derivatives for piezoelectric materials). In this work, therefore, we derive the topological derivative in its closed form for the total potential mechanical energy associated to a thermo-mechanical semi-coupled system, when a small circular inclusion is introduced at an arbitrary point of the domain. In particular, we consider the linear elasticity system (modeled by the Navier equation) coupled with the steady-state heat conduction problem (modeled by the Laplace equation). The mechanical coupling term comes out from the thermal stress induced by the temperature field. Since this term is non-local, we introduce a non-standard adjoint state, which simplifies the analysis allowing to obtain a closed form for the topological derivative. Finally, we provide a full mathematical justification for the derived formula and develop precise estimates for the remainders of the topological asymptotic expansion. We note that this result can be applied in technological research areas such as multi-physic topology design of structures under mechanical and/or thermal loads.

This paper is organized as follows. Section 2 describes the model associated to a thermo-mechanical semi-coupled problem. The topological sensitivity analysis is presented in Section 3, where the main result of this work is derived: the topological derivative in its closed form for the total potential mechanical energy associated to a thermo-mechanical semi-coupled system. Also in this section, a computational framework designed to the numerical validation of the topological derivative formula is presented. The paper ends in Section 4 where concluding remarks are presented.

2. Formulation of the problem

In this work the topological derivative of the total potential energy associated to the mechanical problem submitted to thermal stresses is derived. The topologically perturbed domain is obtained when a small hole is introduced inside the geometrical domain. Then, the resulting void is filled by an inclusion with a contrast on the elastic, thermal and thermal-expansion material properties. Therefore, we need to formulate the problems associated to both original and topologically perturbed domains.

2.1. Unperturbed problem

Consider an open and bounded domain $\Omega \in \mathbb{R}^2$ representing an elastic solid body subject to a linear thermo-mechanical deformation process. Assuming small deformation and variations of temperatures, the functional that represents the total potential energy of the mechanical system for a given temperature field θ is written as:

$$\mathcal{J}_\chi(u,\theta) := \frac{1}{2}\int_\Omega \sigma(u)\cdot\nabla u^s - \int_\Omega Q(\theta)\cdot\nabla u^s - \int_{\Gamma_{N_u}} \bar{t}\cdot u, \tag{5}$$

where u represents the displacement field and $\bar{t}$ is a external traction acting on boundary Γ_{N_u}. The displacement field on the boundary Γ_{D_u} satisfies $u|_{\Gamma_{D_u}} = \bar{u}$, being $\bar{u}$ a prescribed displacement. Moreover, note that $\Gamma_{D_u} \cap \Gamma_{N_u} = \varnothing$ and $\overline{\Gamma_{D_u}} \cup \overline{\Gamma_{N_u}} = \partial\Omega$. The Cauchy stress tensor $\sigma(u)$ in (5) is defined as:

$$\sigma(u) := \mathbb{C}\nabla u^s, \tag{6}$$

where ∇u^s is used to denote the symmetric part of the gradient of the displacement field u, i.e.

$$\nabla u^s := \frac{1}{2}(\nabla u + (\nabla u)^\top). \tag{7}$$

The induced thermal stress tensor $Q(\theta)$ in (5) is defined as:

$$Q(\theta) := \mathbb{C}B\theta. \tag{8}$$

Therefore the total stress, i.e. the contribution of the mechanical and thermal stresses, is defined as

$$S(u,\theta) = \sigma(u) - Q(\theta). \tag{9}$$

In addition, $\mathbb{C}$ denotes the four-order elastic tensor and B denotes the second-order thermo-elastic tensor. In the case of isotropic elastic body, theses tensors are given by:

$$\mathbb{C} = 2\mu II + \lambda(I\otimes I) \quad \text{and} \quad B = \alpha I \Rightarrow \mathbb{C}B = 2\alpha(\lambda+\mu)I, \tag{10}$$

with μ and λ denoting the Lame's coefficients, and α the thermal expansion coefficient. In terms of the engineering constant E (Young's modulus) and ν (Poisson's ratio) the above constitutive response can be written as:

$$\mathbb{C} = \frac{E}{1-\nu^2}[(1-\nu)II + \nu(I\otimes I)] \quad \text{and} \quad \mathbb{C}B = \frac{\alpha E}{1-\nu}I. \tag{11}$$

Considering the previous definitions, we have that the field u is the solution of the following variational problem: given the temperature field θ, find $u \in \mathcal{U}^M$, such that

$$\int_\Omega \sigma(u)\cdot\nabla\eta^s = \int_\Omega Q(\theta)\cdot\nabla\eta^s + \int_{\Gamma_{N_u}} \bar{t}\cdot\eta \quad \forall \eta \in \mathcal{V}^M. \tag{12}$$

In the variational problem (12), the set $\mathcal{U}^M$ and the space $\mathcal{V}^M$ are defined as