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Integral identities for a semi-infinite interfacial crack in anisotropic elastic bimaterials

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ABSTRACT

The focus of the article is on the analysis of a semi-infinite crack at the interface between two dissimilar anisotropic elastic materials, loaded by a general asymmetrical system of forces acting on the crack faces. Recently derived symmetric and skew-symmetric weight function matrices are introduced for both plane strain and antiplane shear cracks, and used together with the fundamental reciprocal identity (Betti formula) in order to formulate the elastic fracture problem in terms of singular integral equations relating the applied loading and the resulting crack opening. The proposed compact formulation can be used to solve many problems in linear elastic fracture mechanics (for example various classic crack problems in homogeneous and heterogeneous anisotropic media, as piezoceramics or composite materials). This formulation is also fundamental in many multifield theories, where the elastic problem is coupled with other concurrent physical phenomena.

1. Introduction and formulation of the problem

The method of singular integral equations in linear elasticity was first developed for solving two-dimensional problems (Muskelishvili, 1953), and later extended to three-dimensional cases by means of multi-dimensional singular integral operators theory (Kupradze et al., 1979; Mikhlin and Prössdorf, 1980). Singular integral formulations for both two and three-dimensional crack problems have been derived by means of a general approach based on Green's function method (Weaver, 1977; Budiansky and Rice, 1979; Linkov et al., 1997). As a result, the displacements and the stresses are defined by integral relations involving the Green's functions, for which explicit expressions are required (Bigoni and Capuani, 2002). Although Green's functions for many two and three-dimensional crack problems in isotropic and anisotropic elastic materials have been derived (Sinclair and Hirth, 1975; Weaver, 1977; Pan, 2000, 2003; Pan and Yuan, 2000), their utilization in evaluating physical displacements and stress fields on the crack faces implies, especially in the anisotropic case, challenging numerical estimation of integrals which convergence should be asserted carefully. Moreover, the approach based on Green's function method works when the tractions applied on the discontinuity surface are symmetric, but not in the case of asymmetric loading acting on the crack faces.

Recently, using a procedure based on Betti's reciprocal theorem and weight functions[1] an alternative method for deriving integral identities relating the applied loading and the resulting crack opening has been developed for two and three-dimensional semi-infinite interfacial cracks between dissimilar isotropic materials by Piccolroaz and Mishuris (2013). In the two-dimensional case, the obtained identities contain Cauchy type singular operators together with algebraic terms. The algebraic terms vanish in the case of homogeneous materials. This approach avoids the use of the Green's functions without assuming the load to be symmetric.

The aim of this paper is to derive analogous integral identities for the case of semi-infinite interfacial cracks in anisotropic bimaterials subjected to two-dimensional deformations.

General expressions for symmetric and skew-symmetric weight functions for interfacial cracks in two-dimensional anisotropic bimaterials have been recently derived by Morini et al. (2013) by means of Stroh representation of displacements and fields (Stroh, 1962) combined with a Riemann–Hilbert formulation of the traction problem at the interface (Suo, 1990b). These expressions for the weight functions are used together with the results obtained for isotropic media by Piccolroaz and Mishuris (2013) in order to obtain integral formulation for interfacial cracks problems in

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[1] Defined by Bueckner (1985) as singular non-trivial solutions of the homogeneous traction-free problem and later derived for general three-dimensional problems by Willis and Movchan (1995), and for interfacial cracks by Gao (1992) and Piccolroaz et al. (2009).

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anisotropic bimaterial solids with general asymmetric load applied at the crack faces.

We consider a two-dimensional semi-infinite crack between two dissimilar anisotropic elastic materials with asymmetric loading applied to the crack faces, the geometry of the system is shown in Fig. 1. Further in the text, we will use the superscripts $^{(1)}$ and $^{(2)}$ to denote the quantities related to the upper and the lower elastic half planes, respectively. The crack is situated along the negative semi-axis $x_1 < 0$. Both in-plane and antiplane stress and deformation, which in fully anisotropic materials are coupled (Ting, 1995), are taken into account. The symmetrical and skew-symmetrical parts of the loading are defined as follows:

$$\langle \mathbf{p} \rangle = \frac{1}{2}(\mathbf{p}^+ + \mathbf{p}^-), \quad [\![\mathbf{p}]\!] = \mathbf{p}^+ - \mathbf{p}^-, \tag{1}$$

where $\mathbf{p}^+$ and $\mathbf{p}^-$ denote the loading applied on the upper and lower crack faces, $x_2 = 0^+$ and $x_2 = 0^-$, respectively (see Fig. 1).

In Section 2 preliminary results needed for the derivation of the integral identities and for the complete explanation of the proposed method are reported. In Section 2.1, the fundamental reciprocal identity and the weight functions, defined as special singular solution of the homogeneous traction-free problem are introduced. In Section 2.2, symmetric and skew-symmetric weight functions matrices for interfacial cracks in anisotropic bimaterials recently derived by Morini et al. (2013) are reported.

Section 3 contains the main results of the paper: integral identities (34), (35), (64) and (65) for two-dimensional crack problems between two dissimilar anisotropic materials are derived and discussed in details. The integral identities are derived for monoclinic-type materials, which are the most general class of anisotropic media where both in-plane and antiplane strain and in-plane and antiplane stress are uncoupled (Ting, 1995; Ting, 2000), and the Mode III can be treated separately by Mode I and II. By means of Betti's formula and weight functions, both antiplane and plane strain fracture problems are formulated in terms of singular integral equations relating the applied loading and the resulting crack opening.

In Section 4, the obtained integral identities are used for studying cracks in monoclinic bimaterials loaded by systems of line forces acting on the crack faces. The proposed examples show that using the identities explicit expressions for crack opening and tractions ahead of the crack tip can be derived for both antiplane and in-plane problems. These simple illustrative cases demonstrate also that the proposed integral formulation is particularly easy to apply and can be very useful especially in the analysis of phenomena where the elastic behavior of the material is coupled with other physical effects, as for example hydraulic fracturing, where both anisotropy of the geological materials and fluid motion must be taken into account.

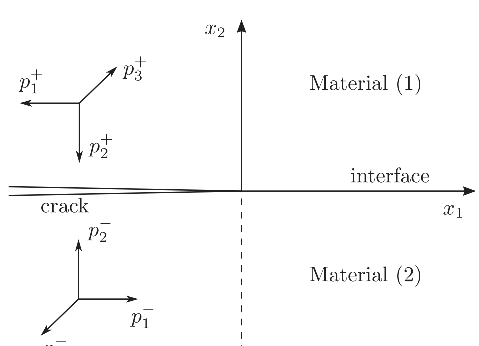

Fig. 1. Two-dimensional semi-infinite interfacial crack loaded by non necessarily symmetric forces applied on the crack faces.

Finally, in Appendix A, the Stroh formalism (Stroh, 1962), adopted by Suo (1990b) and Gao et al. (1992) in analysis of interfacial cracks in anisotropic bimaterials and recently used by Morini et al. (2013) for deriving symmetric and skew-symmetric weight functions, is briefly explained. In particular, explicit expressions for Stroh matrices and surface admittance tensor needed in weight functions expressions associated to monoclinic materials are reported.

2. Preliminary results

In this section relevant results obtained by several studies regarding interfacial cracks are reported. These results will be used further in the paper in order to develop an integral formulation for the problem of a semi-infinite interfacial crack in anisotropic bimaterials.

In Section 2.1, we introduce the Betti integral formula for a crack in an elastic body subjected to two-dimensional deformations with general asymmetric loading applied at the faces.

In Section 2.2, general matrix equations expressing weight functions in terms of the associate singular traction vectors, recently derived by Morini et al. (2013), and valid for interfacial cracks in a wide range of two-dimensional anisotropic bimaterials are reported.

2.1. The Betti formula

The Betti formula is generally used in linear elasticity in order to relate the physical solution to the weight function which is defined as special singular solution to the homogeneous traction-free problem (Bueckner, 1985; Willis and Movchan, 1995). Since the Betti integral theorem is independent of the specific elastic constitutive relations of the material, it applies to both isotropic and anisotropic media in the same form.

The notations $\mathbf{u} = (u_1, u_2, u_3)^T$ and $\boldsymbol{\sigma} = (\sigma_{21}, \sigma_{22}, \sigma_{23})^T$ are introduced to indicate respectively the physical displacements and the traction vector acting on the plane $x_2 = 0$. According to the fact that two-dimensional elastic deformations are here considered, both displacements and stress do not depend on the variable x_3. Nevertheless, since both in-plane and anti-plane strain and stress are considered, non-zero components u_3 and σ_{23} are accounted for (Ting, 1995). The notations $\mathbf{U} = (U_1, U_2, U_3)^T$ and $\boldsymbol{\Sigma} = (\Sigma_{21}, \Sigma_{22}, \Sigma_{23})^T$ are introduced to indicate the weight function, defined by Bueckner (1985) as a non-trivial singular solution of the homogeneous traction-free problem, and the associated traction vector, respectively. As it was shown by Willis and Movchan (1995), the weight function $\mathbf{U}$ is defined in a different domain respect to physical displacement, where the crack is placed along the positive semi-axis $x_2 > 0$. Following the procedure reported and discussed in Willis and Movchan (1995), Piccolroaz et al. (2009) and Piccolroaz and Mishuris (2013), from the application of the Betti integral formula to the physical fields and to weight functions for both the upper and the lower half-planes in Fig. 1, we obtain:

$$\begin{aligned}\int_{-\infty}^{\infty} \{ & \tilde{\mathbf{R}}\mathbf{U}(x_1' - x_1, 0^+) \cdot \mathbf{p}^+(x_1) - \tilde{\mathbf{R}}\mathbf{U}(x_1' - x_1, 0^-) \cdot \mathbf{p}^-(x_1) \\ & + \tilde{\mathbf{R}}\mathbf{U}(x_1' - x_1, 0^+) \cdot \boldsymbol{\sigma}^{(+)}(x_1, 0^+) - \tilde{\mathbf{R}}\mathbf{U}(x_1' - x_1, 0^-) \cdot \boldsymbol{\sigma}^{(+)}(x_1, 0^-) \\ & - [\tilde{\mathbf{R}}\boldsymbol{\Sigma}(x_1' - x_1, 0^+) \cdot \mathbf{u}(x_1, 0^+) + \tilde{\mathbf{R}}\boldsymbol{\Sigma}(x_1' - x_1, 0^-) \cdot \mathbf{u}(x_1, 0^-)]\} dx_1 = 0,\end{aligned} \tag{2}$$

where $\tilde{\mathbf{R}}$ is the rotation matrix:

$$\tilde{\mathbf{R}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$