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Strain gradient solution for a finite-domain Eshelby-type plane strain inclusion problem and Eshelby's tensor for a cylindrical inclusion in a finite elastic matrix

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ABSTRACT

A solution for the finite-domain Eshelby-type inclusion problem of a finite elastic body containing a plane strain inclusion prescribed with a uniform eigenstrain and a uniform eigenstrain gradient is derived in a general form using a simplified strain gradient elasticity theory (SSGET). The formulation is facilitated by an extended Betti's reciprocal theorem and an extended Somigliana's identity based on the SSGET and suitable for plane strain problems. The disturbed displacement field is obtained in terms of the SSGET-based Green's function for an infinite plane strain elastic body, which differs from that in earlier studies using the three-dimensional Green's function. The solution reduces to that of the infinite-domain inclusion problem when the boundary effect is suppressed. The problem of a cylindrical inclusion embedded concentrically in a finite plane strain cylindrical elastic matrix of an enhanced continuum is analytically solved for the first time by applying the general solution, with the Eshelby tensor and its average over the circular cross section of the inclusion obtained in closed forms. This Eshelby tensor, being dependent on the position, inclusion size, matrix size, and a material length scale parameter, captures the inclusion size and boundary effects, unlike existing ones. It reduces to the classical elasticity-based Eshelby tensor for the cylindrical inclusion in an infinite matrix if both the strain gradient and boundary effects are not considered. Numerical results quantitatively show that the inclusion size effect can be quite large when the inclusion is very small and that the boundary effect can dominate when the inclusion volume fraction is very high. However, the inclusion size effect is diminishing with the increase of the inclusion size, and the boundary effect is vanishing as the inclusion volume fraction becomes sufficiently low.

1. Introduction

Eshelby's eigenstrain method and fourth-order strain transformation tensor (Eshelby, 1957, 1959) play a key role in homogenization methods for heterogeneous materials (e.g., Hill, 1965; Budiansky, 1965; Mori and Tanaka, 1973; Weng, 1990; Huang et al., 1994; Le Quang and He, 2007; Genin and Birman, 2009). However, the Eshelby tensor in its original form (Eshelby, 1957, 1959) is based on *classical elasticity* and cannot account for the particle (inclusion) size effect experimentally observed in some composites filled with micro- and nano-particles (e.g., Vollenberg and Heikens, 1989; Reynaud et al., 2001; Cho et al., 2006). Moreover, this classical Eshelby tensor is for an inclusion embedded in an *infinite* elastic matrix and is unable to incorporate the effect of finite boundaries. As a result, the homogenization methods employing the classical elasticity-based Eshelby tensor cannot capture the particle size and boundary effects. Hence, there has been a need to obtain Eshelby's tensor for an inclusion in a *finite* matrix using *higher-order* (non-classical) elasticity theories, which, unlike classical elasticity, contain material length scale parameters and are capable of explaining microstructure-dependent size (and other) effects.

For the Eshelby-type inclusion problem of an *infinite* homogeneous isotropic elastic body containing an inclusion, a number of studies have been conducted using various higher-order elasticity theories, which include a micropolar theory (Cheng and He, 1995, 1997; Ma and Hu, 2006), a microstretch theory (Liu and Hu, 2004; Kiris and Inan, 2006; Ma and Hu, 2007), a modified couple stress theory (Zheng and Zhao, 2004), a strain gradient theory (Zhang and Sharma, 2005), and a simplified strain gradient theory (Gao and Ma, 2009, 2010a,b; Ma and Gao, 2010a). These studies have led to analytical solutions of the inclusion problem and resulted in closed-form expressions of the Eshelby tensor for a spherical or cylindrical inclusion in an infinite elastic body based on higher-order elasticity theories.

On the other hand, for the problem of an inclusion embedded in a *finite* homogeneous isotropic elastic matrix, only a few analytical studies have been performed even in the context of *classical elasticity*. The first one was provided by Kinoshita and Mura (1984). They

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proved the existence and uniqueness of a second-order Neumann tensor, which reduces to the Green's function (also a second-order tensor) when the body is unbounded. The use of the Neumann tensor would give the solution of an inclusion problem in a bounded elastic body. However, the determination of this Neumann tensor for a bounded elastic body is rather challenging, and only the Neumann tensor for a half space was provided in Kinoshita and Mura (1984). More recently, Li et al. (2005, 2007) analytically obtained the Eshelby's tensors for a two-dimensional (2-D) finite-domain circular inclusion problem and a three-dimensional (3-D) finite-domain spherical inclusion problem using Somigliana's identity and Green's functions in classical elasticity.

The first study on *finite-domain* inclusion problems based on a *higher-order* elasticity theory has recently been reported by Gao and Ma (2010a), where a simplified strain gradient elasticity theory (SSGET) (e.g., Gao and Park, 2007) is used and the problem of a spherical inclusion embedded concentrically in a finite spherical elastic body is analytically solved. The solution of this finite-domain inclusion problem is obtained using the SSGET-based 3-D Green's function derived in Gao and Ma (2009) and includes the solution for its counterpart infinite-domain inclusion problem published earlier as a limiting case.

The current study aims to provide the solution for the finite-domain Eshelby-type inclusion problem of a finite homogeneous isotropic elastic body containing a plane strain inclusion prescribed with a uniform eigenstrain and a uniform eigenstrain gradient using the SSGET. The present solution utilizes the SSGET-based Green's function for a plane strain elastic body, which differs from the 3-D Green's function used in Gao and Ma (2010a) for the finite-domain spherical inclusion problem and in Ma and Gao (2010a) for the infinite-domain plane strain and cylindrical inclusion problems.

The rest of the paper is organized as follows. In Section 2, the SSGET is first reviewed, which is followed by the derivation of a general solution for the finite-domain Eshelby-type plane strain inclusion problem using an extended Betti's reciprocal theorem and an extended Somigliana's identity based on the SSGET and suitable for plane strain problems. The finite-domain cylindrical inclusion problem is solved in Section 3 by applying the general formulas derived in Section 2, which leads to closed-form expressions of the Eshelby tensor and its area average. In Section 4, sample numerical results are presented to quantitatively show the dependence of the components of the Eshelby tensor and its average obtained in Section 3 on the position, inclusion size, and inclusion volume fraction, where the size and boundary effects are observed and discussed. The paper concludes in Section 5 with a summary and some remarks.

2. Solution for a plane strain inclusion in a finite domain

2.1. Simplified strain gradient elasticity theory (SSGET)

The SSGET is the simplest strain gradient elasticity theory evolving from Mindlin's pioneering work (Mindlin, 1964, 1965; Mindlin and Eshel, 1968). It is also known as the first gradient elasticity theory of Helmholtz type and the dipolar gradient elasticity theory (Gao and Ma, 2010a). According to this theory, the strain energy density function, w, for an isotropic linearly elastic material has the form (e.g., Gao and Park, 2007; Gao and Ma, 2010b):

$$w = w(\varepsilon_{ij}, \kappa_{ijk}) = \frac{1}{2}\lambda\varepsilon_{ii}\varepsilon_{jj} + \mu\varepsilon_{ij}\varepsilon_{ij} + L^2\left(\frac{1}{2}\lambda\kappa_{iik}\kappa_{jjk} + \mu\kappa_{ijk}\kappa_{ijk}\right), \quad (1)$$

where λ and μ are the Lamé constants in classical elasticity, L is a material length scale parameter, and ε_{ij} and κ_{ijk} are, respectively, the components of the infinitesimal strain, $\boldsymbol{\varepsilon} = \varepsilon_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, and the strain gradient, $\boldsymbol{\kappa} \equiv \nabla\boldsymbol{\varepsilon} = \kappa_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$, given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \kappa_{ijk} \equiv \varepsilon_{ij,k} = \frac{1}{2}(u_{i,jk} + u_{j,ik}), \quad (2a,b)$$

with u_i being the components of the displacement vector $\mathbf{u} = u_i\mathbf{e}_i$.

The constitutive equations are obtained from Eq. (1) as

$$\tau_{ij} = \frac{\partial w}{\partial \varepsilon_{ij}} = \lambda\varepsilon_{ll}\delta_{ij} + 2\mu\varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} = \tau_{ji}, \quad (3)$$

$$\mu_{ijk} = \frac{\partial w}{\partial \kappa_{ijk}} = L^2\left(\lambda\varepsilon_{ll}\delta_{ij} + 2\mu\varepsilon_{ij}\right)_{,k} = L^2 C_{ijmn}\kappa_{mnk} = L^2\tau_{ij,k} = \mu_{jik}, \quad (4)$$

where τ_{ij} are the components of the Cauchy stress, $\boldsymbol{\tau} = \tau_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, μ_{ijk} are the components of the double stress, $\boldsymbol{\mu} = \mu_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$, δ_{ij} is the Kronecker delta, and C_{ijkl} are the components of the elastic stiffness tensor for isotropic elastic materials given by $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$.

The equilibrium equations are

$$\sigma_{ij,j} + f_i = 0, \quad (5)$$

where f_i are the components of the body force, and σ_{ij} are the components of the total stress, $\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, which are related to the Cauchy stress components τ_{ij} through

$$\sigma_{ij} \equiv \tau_{ij} - \mu_{ijk,k} = \tau_{ij} - L^2\tau_{ij,kk}. \quad (6)$$

Using Eqs. (2a,b)–(4) and (6) in Eq. (5) leads to the Navier-like displacement-equations of equilibrium as

$$(\lambda + \mu)u_{i,ij} + \mu u_{j,kk} - L^2\left[(\lambda + \mu)u_{i,ij} + \mu u_{j,kk}\right]_{,mm} + f_j = 0 \quad \text{in } \Omega, \quad (7)$$

where Ω is the region occupied by the elastic material.

The complete boundary conditions, determined simultaneously with the equilibrium equations listed in Eq. (5) using a variational formulation (Gao and Park, 2007), have the form:

$$\left.\begin{array}{l} t_i = \bar{t}_i \quad \text{or} \quad u_i = \bar{u}_i, \\ q_i = \bar{q}_i \quad \text{or} \quad u_{i,l}n_l = \frac{\partial \bar{u}_i}{\partial n} \end{array}\right\} \quad \text{on } \partial\Omega, \quad (8a,b)$$

with

$$t_i = \sigma_{ij}n_j - \left(\mu_{ijk}n_k\right)_{,j} + \left(\mu_{ijk}n_k n_l\right)_{,l} n_j, \quad q_i = \mu_{ijk}n_j n_k, \quad (8c,d)$$

where t_i and q_i are, respectively, the components of the Cauchy traction vector and double stress traction vector, $\partial\Omega$ is the smooth bounding surface of Ω, and n_i is the outward unit normal vector on $\partial\Omega$. In Eqs. (8a,b), the overbar represents the prescribed value. Note that the standard index notation, together with the Einstein summation convention, is used in Eqs. (1)–(8a–d) and throughout this paper, with each Latin index (subscript) ranging from 1 to 3 and each Greek index ranging from 1 to 2, unless otherwise stated.

Eqs. (7) and (8a,b), along with Eqs. (2a,b)–(4) and (6), define the boundary value problem in terms of displacement in the SSGET. Clearly, the material length scale parameter L is explicitly involved in Eq. (7) in addition to the two Lamé constants λ and μ. When the strain gradient effect is absent (i.e., $L = 0$), it follows from Eq. (4) that $\mu_{ijk} = 0$ and from Eq. (6) that $\sigma_{ij} = \tau_{ij}$. As a result, Eqs. (7) and (8a,b) reduce to the governing equations and the boundary conditions in terms of displacement in classical elasticity (e.g., Timoshenko and Goodier, 1970; Gao and Rowlands, 2000).

For an infinite elastic body loaded by a unit concentrated force, Eq. (7), subject to the boundary conditions of $\mathbf{u}$ and its first-, second- and third-order spatial derivatives vanishing at infinity, has been solved in Gao and Ma (2009) by using Fourier transforms to obtain the SSGET-based 3-D Green's function expressed in terms of elementary functions. This Green's function has been subsequently used to solve several inclusion problems involving an infinite or a finite 3-D elastic body containing an inclusion (Gao and Ma, 2009, 2010a,b; Ma and Gao, 2010a).