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# Elastic buckling of columns with end restraint effects

R. Adman [a], M. Saidani [b,*]

[a] Faculty of Civil Engineering, U.S.T.H.B., Algiers, Algeria
[b] Faculty of Engineering and Computing, Coventry University, Priory Street, Coventry CV1 5FB, UK

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ABSTRACT

It is a well established fact that the behaviour of columns as part of a structure is affected by the end restraints. The main aim of the current study is to develop a criterion of stability capable of predicting an impending failure by elastic buckling of a column of a structure. The rigidities at the ends of a column element are modelled using rotational and translational springs, which have been considered by taking into account their coupling effects. The role of the springs is to model the nodal restraints of any column of a given structure. This formulation offers significant practical advantages in the elastic buckling analysis of such structures. This approach is performed through a relationship to several parameters, such as the non-dimensional rotational and translational restraint indices and the effective length factor K. The approach was applied in analysing the elastic buckling of a number of structures and good results were obtained, thus justifying its reliability. In determining the effective length factor K, a marked difference was noted between the results obtained using the Eurocode approach and that proposed by the current study, particularly in the case of non-braced structures.


## 1. Introduction and context of the current work

A substantial body of research has been carried out by a number of researchers on the stability of frames and the concept of effective length and effective length factors. In the 70s, Wood [1–3] investigated the effective length of columns in multi-storey buildings. In doing so, he defined the rigidity of a joint in a multi-storey setting in terms of the effective length factor K. The Eurocode later adopted this approach. This approach, although limited in practice, remains a powerful analytical tool for the engineer [4,5]. Cheong-Siat-Moy [6] and Chen and Lui [7] have developed expressions for the effective length factor for single columns with partial lateral end restraints. However, the effect of coupling between rotational and translational rigidities is not taken into account. Also the relative stiffness G is simply adopted as given in the codes.

The K factor represents an important parameter vis-à-vis the elastic buckling analysis. It can easily accommodate the elastic critical load by using a single formula covering all situations of boundary condition, expressed by the following equation:

$$N = \pi^2 EI/(KL)^2 \quad (1)$$

where, K represents the ratio between the effective length "$L_f$" and the actual length "L" of the column:

$$K = L_f/L \quad (2)$$

From a physical point of view, "the effective length is the length of the equivalent pin-ended column that would have the same elastic critical load as the actual end-restrained column" [8].

This study offers a simple and yet a global approach that allows a very rigorous assessment of the effective length factor K. The criterion of analysis derives from the solution of the deflection y(x) of a column element where four springs are introduced at the ends to model the rotational and translational flexibilities. This formulation offers significant practical advantages in the elastic buckling analysis of such structures. Furthermore, the values of the effective length factor K for some situations of conventional boundary condition are known. Fig. 1 below provides an overview regarding the values of K and their respective buckling modes. Obviously, in practice there are an infinite number of situations relating to the boundary conditions, which do not correspond to conventional situations especially when the column element of interest is considered in relation to partially braced structures.

Hellesland and Bjorhovde [9] used what they described the method of means to determine the effective length factors for continuous columns and beams. The method uses effective length factors from isolated columns as input. Comparison between actual results and those using the method were found to be in order of 5%. The method was limited when dealing with unbraced frames or when columns were laterally very flexible. Also, coupling effects were not considered.

Previous researchers such as Aristizabal-Ochoa [10,11] and Hellesland [12,13] used the so-called fixity factors or as also known degree of rotational fixity factor in solving the stability equations. However, such factors do not take into account the coupling effects between rotational and translational flexibilities.

* Corresponding author. Tel./fax: +44 2476888385.
*E-mail address:* m.saidani@coventry.ac.uk (M. Saidani).

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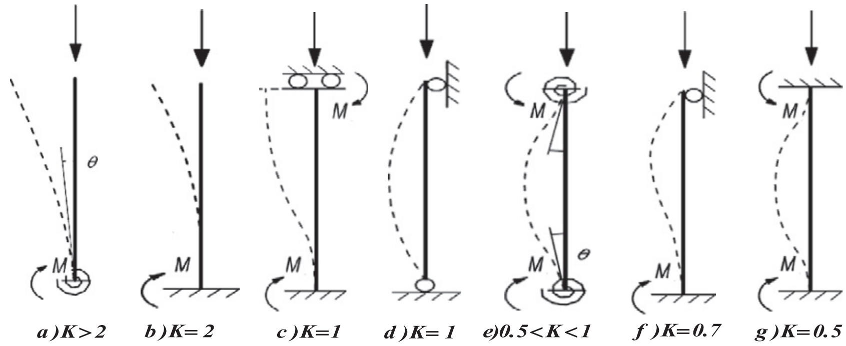


**Fig. 1.** Values of K and respective buckling modes for some conventional boundary conditions.

The novelty of the current study is that efficient and more general parameters, equivalent to those previously cited, were obtained by solving the problem of global stability of the column with end restraints with coupling effects fully taken into consideration.

The purpose of the current study is to investigate the effect of end restraints on the column elastic buckling. This is achieved by proposing a general criterion resulting from the solution of the stability equations, taking into account the end conditions of the column in terms of non-dimensional translational and rotational end restraint indices and their coupling effects.

## 2. Formulation

Consider a column segment (ij) given in Fig. 2.

The equilibrium equation along the element length can be expressed in compressive case as:

$$y(x)_{,xx} + (\beta/L)^2 y(x) = -M(x)/EI \tag{3}$$

where the non-dimensional parameter $\beta$ is given as:

$$\beta = \sqrt{\frac{|N|L^2}{EI}} \tag{4}$$

and (,xx) indicate the second derivative operator.

$$M(x) = M_i + T_i \cdot x \tag{5}$$

$$T_i = \left(M_j - M_i\right)/L \tag{6}$$

N is the axial force, $M_i$ and $M_j$ are the nodal moments, $T_i$ and $T_j$ are the nodal shear forces (shown as positives in Fig. 1), E = Young modulus and I = second moment of area of the section about the axis of bending.

The solution of Eq. (3) is obtained as:

$$y(x) = C_1 \cos\left(\beta \frac{x}{L}\right) + C_2 \sin\left(\beta \frac{x}{L}\right) + C_3\left(\frac{x}{L}\right) + C_4 \tag{7}$$

$C_1$, $C_2$, $C_3$, and $C_4$ are constants depending on the boundary conditions of the element (ij). Four degrees of freedom (d.o.f.) are taken into account: two displacements $v_i$ and $v_j$ according to oy axis, and two rotations $\theta_i$ and $\theta_j$ around oz axis, (⊥xoy).

To undertake this task, it is essential to consider the exact boundary conditions on the ends of the nodal element. Indeed, the nodal displacements and rotations described by the variables ($v_i$ $\theta_i$, $v_j$, $\theta_j$), depend directly of requirements for the ends of the element. In this regard, the physical model illustrated in Fig. 3 is adopted, which is distinguished by an unconventional behaviour at each node of the element.

The particular solution for Eq. (3), which refers to the loading applied to the column has no effect on the formulation of the stiffness matrix, and therefore has no influence on the parameter K being the main objective of this study.

By using the appropriate boundary conditions,

$$\begin{cases} v_i = y(0) + v_i^y & \text{(a)} \\ \theta_i = y_{,x}(0) + \phi_i^r & \text{(b)} \\ v_j = y(L) + v_j^y & \text{(c)} \\ \theta_j = y_{,x}(L) + \phi_j^r & \text{(d)} \end{cases} \tag{8}$$

where, ($\phi_i^r$,$\phi_j^r$,$v_i^y$,$v_j^y$) refers to Eq. (8(a), (b), (c), and (d)), the rotations and displacements, respectively associated with the springs of rotation

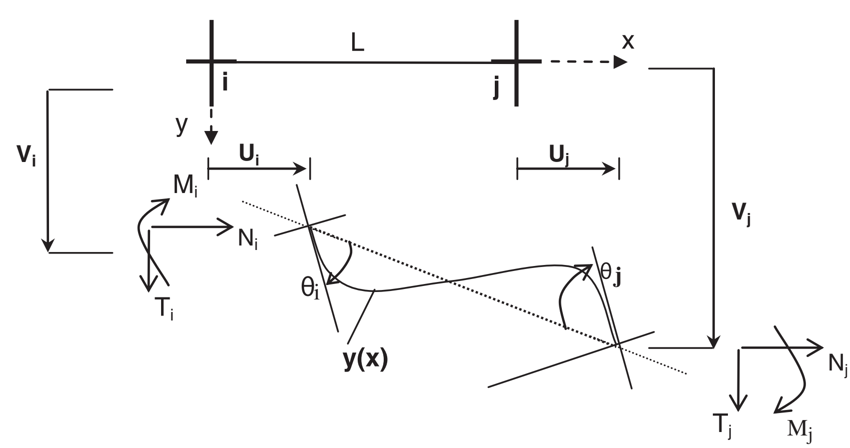


**Fig. 2.** Column model (end moments, forces, rotations and deflections).