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Exceptional points in the thermoacoustic spectrum

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ABSTRACT

Exceptional points are found in the spectrum of a prototypical thermoacoustic system as the parameters of the flame transfer function are varied. At these points, two eigenvalues and the associated eigenfunctions coalesce. The system's sensitivity to changes in the parameters becomes infinite. Two eigenvalue branches collide at the exceptional point as the interaction index is increased. One branch originates from a purely acoustic mode, whereas the other branch originates from an intrinsic thermoacoustic mode. The existence of exceptional points in thermoacoustic systems has implications for physical understanding, computing, modeling and control.

1. Introduction

At exceptional points (EPs), at least two eigenvalues and the associated eigenfunctions coalesce, and the eigenvalue sensitivity with respect to changes in the parameters becomes infinite [1,2]. Interesting physical phenomena associated with EPs appear across various disciplines from quantum mechanics through optics and acoustics [2–4]. To the best of the authors' knowledge, the role of exceptional points has not yet been explored in thermoacoustic systems, although points in the parameter space with infinite sensitivity were discussed in a recent review article [5]. In this letter, we show that these points in the thermoacoustic spectrum are exceptional, and that they can be found in a generic thermoacoustic system when two real parameters are varied.

1.1. Thermoacoustic instabilities

Thermoacoustic instabilities are a major challenge for the reliable operation of many technical combustion systems, as reviewed by Ref. [5] and references therein. For most practical applications with low-Mach number combustion, thermoacoustic phenomena can be modelled by an inhomogeneous Helmholtz equation, which reads

$$\nabla \cdot \left(\bar{c}^2 \nabla \hat{p}\right) + \omega^2 \hat{p} = -\mathrm{i}\omega(\gamma - 1)\hat{q}, \tag{1}$$

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where ω is the complex frequency, $\bar{c}$ is the mean speed of sound, $\mathrm{i}^2 = -1$, and γ is the heat-capacity ratio. $\hat{p}$ and $\hat{q}$ are the Fourier-transformed fluctuations[1] of acoustic pressure and heat release rate, respectively. Quantities are non-dimensionalized with a characteristic length, speed of sound, and density. The heat release rate fluctuation is commonly related to a velocity fluctuation at a reference position by a time-delay model [5], i.e. $-\mathrm{i}\omega(\gamma-1)\hat{q} = n\exp(-\mathrm{i}\omega\tau)\nabla\hat{p}\,|_{x_{\mathrm{ref}}}$, where the parameters n and τ are the interaction index and the time delay, respectively. The thermoacoustic stability problem is generally non-Hermitian because of the flame response term and dissipative boundary conditions. On numerical discretization or travelling-wave decomposition [5], thermoacoustic stability is governed by a nonlinear eigenvalue problem [6,7]

$$\mathbf{L}(\omega;\boldsymbol{\varepsilon})\hat{\boldsymbol{p}} = 0, \tag{2}$$

where the vector $\boldsymbol{\varepsilon} \in \mathbb{R}^M$ contains M parameters related to, for example, the mean speed of sound, the geometry, the flame response, and the boundary conditions. $\mathbf{L} \in \mathbb{C}^{N\times N}$ is an analytic function of $\boldsymbol{\varepsilon}$ and ω in some subdomain of $\mathbb{R}^M \times \mathbb{C}$, where N is the number of degrees of freedom of the discretized equations. For a given $\boldsymbol{\varepsilon}$, the stability of the linear system is characterized by the eigenvalues $\omega = \omega_r + \mathrm{i}\omega_i$, where $\omega_r \in \mathbb{R}$ is the angular frequency and $-\omega_i \in \mathbb{R}$ is the growth rate of the linear oscillation. With this convention, the system is linearly stable if $\omega_i > 0$. The associated thermoacoustic mode shapes are provided by the eigenvectors $\hat{\boldsymbol{p}} \in \mathbb{C}^N$.

1.2. Eigenvalue classification

Eigenvalues can be classified according to their algebraic and geometric multiplicities, a and g. The algebraic multiplicity is the eigenvalue's multiplicity as a root of the dispersion relation, whereas the geometric multiplicity is the dimension of the associated eigenspace, i.e. the number of linearly independent eigenvectors. An eigenvalue of (2) can be either semi-simple, when $a = g$; or defective, when $a > g$. For the special case $a = g = 1$ an eigenvalue is called simple. Semi-simple eigenvalues with $g > 1$ and defective eigenvalues are referred to as degenerate eigenvalues. Defective eigenvalues that are branch-point singularities in the parameter space are called exceptional points (EPs). Eigenvalues of single-flame longitudinal thermoacoustic systems are typically simple [5,8]. Systems with discrete rotational symmetry, such as annular and can-annular combustors, feature semi-simple degenerate eigenvalues [6,9], with fewer simple eigenvalues.

1.3. Sensitivity at an exceptional point

Mathematically, in the neighborhood of an EP, the eigenvalue has a perturbation expansion in fractional powers of the parameter (Section II-2.2 in Ref. [1]), also known as Puiseux series. At an EP with $a = 2$ (hence $g = 1$), which is assumed in the remainder of this letter, the change of the eigenvalue due to a perturbation to the i-th parameter, ε_i, reads

$$\omega = \omega_{\mathrm{EP}} + \omega_1\sqrt{\varepsilon_i - \varepsilon_{i,\mathrm{EP}}} + O\left(\varepsilon_i - \varepsilon_{i,\mathrm{EP}}\right), \qquad \varepsilon_i \to \varepsilon_{i,\mathrm{EP}}, \tag{3}$$

where ω_1 is a constant. Thus, the first-order sensitivity $\partial\omega/\partial\varepsilon_i|_{\omega_{\mathrm{EP}},\boldsymbol{\varepsilon}_{\mathrm{EP}}}$ with respect to any parameter, ε_i, is infinite[2] [2] because $(\omega - \omega_{\mathrm{EP}})/(\varepsilon_i - \varepsilon_{i,\mathrm{EP}}) \to \infty$ as $\varepsilon_i \to \varepsilon_{i,\mathrm{EP}}$. An equivalent expansion holds for the eigenfunction at the EP.

1.4. Calculation of exceptional points in thermoacoustics

We consider a thermoacoustic system with an $n-\tau$ flame model and calculate EPs as n and τ are varied. The eigenvalues are the roots of the dispersion relation

$$D(\omega; n, \tau) = 0, \tag{4}$$

where $D(\omega;n,\tau) \equiv \det[\mathbf{L}(\omega;n,\tau)]$ is the characteristic function, which is transcendental and analytic in ω in some subdomain of the complex plane. For an eigenvalue to have $a = 2$, (4) must be satisfied with the two following conditions

$$\frac{\partial D}{\partial\omega}(\omega;n,\tau) = 0, \tag{5}$$

$$\frac{\partial^2 D}{\partial\omega^2}(\omega;n,\tau) \neq 0. \tag{6}$$

The solution of the two complex-valued equations (4) and (5) is the set of parameters (n_{EP}, τ_{EP}) and the defective eigenvalue ω_{EP}. Equations (4) and (5) would also be satisfied for degenerate semi-simple eigenvalues, such as those found in systems with rotational symmetry. However, in systems without symmetry, which we consider here, degenerate eigenvalues are generically defective [10]. The defective eigenvalue has algebraic multiplicity two, but there is only one associated eigenvector $\hat{\boldsymbol{p}}_{\mathrm{EP}}$.

[1] e.g., a fluctuation evolves as $\hat{(\cdot)}\exp(\mathrm{i}\omega t)$.

[2] This is in contrast to the semi-simple case, in which the first-order sensitivity is finite (Theorem II-2.3 in Ref. [1]).