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Reduced-order modelling of wave propagation in an elastic layer of constant curvature and thickness

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ABSTRACT

This paper is concerned with reduced-order modelling of wave propagation in an elastic layer of constant curvature and thickness by means of the generalised Galerkin method with Legendre polynomials used as coordinate functions. A new family of polynomial approximations to the dispersion relation and corresponding approximations to the field variables are obtained. These approximations have high accuracy, particularly in resolving the surface waves which are dominant features of the solution. The convergence rate is assessed by alternative accuracy measures and shown to be exponentially fast while the order of polynomials increases at a slow and regular rate. Detailed analysis of displacements and stresses in (frequency, wavenumber) space is performed. This novel modelling should facilitate studies of mode conversion around bends, where short waves are involved, for example in soft materials.

1. Introduction

The analysis of wave propagation in an elastic layer of constant curvature and thickness is an ideal benchmark problem to assess accuracy and efficiency of various approximate theories and methods. The exact solutions in the plane strain case are readily available for a straight layer [1,2] and for a 'thick-walled hollow cylinder' [3]. The theories, or models, of Bernoulli-Euler, Mindlin-Herrmann and Timoshenko are generally recognized as 'low frequency – long wave' approximations of the exact Rayleigh-Lamb solution. The thin shell approximation referred to in the literature as a curved beam theory [2,4,5] is also known as a 'low frequency – long wave' approximation of the exact solution of the problem for a layer of constant curvature.

Recently, a hierarchy of reduced-order models of elastic wave propagation in a straight layer have been formulated in Refs. [6,7] to capture as many branches of dispersion diagram as necessary. The two distinctive features of these models are that the dispersion equations are formulated as low-order polynomials in both frequency and wavenumber and that the cut-on frequencies match their exact counterparts. The idea of the present paper is to formulate similar hierarchy for a layer of constant curvature and to assess ranges of validity of its members. However, the methodology employed here differs profoundly from those used in Refs. [6,7]. The point of departure is the energy functional, and the variational method, often referred to as the generalised Galerkin's method [8,9] is used with the Legendre polynomials as the coordinate functions. By these means, the governing differential equations, which provide a polynomial dispersion equation at any approximation level, are derived. Although this methodology is described both in the context of alternative projection methods [10] and in

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the context of the finite element method [11, Chapter 3], it has not yet been, to the best of our knowledge, used for analysis of elastic wave propagation.

The derivation of governing equations is presented in Section 2. The alternative convergence and accuracy measures are introduced and discussed in Section 3. Section 4 is concerned with analysis of dispersion diagrams, obtained at different approximation levels. The detailed field analysis and convergence studies are presented in Section 5. Results of studies are summarized in Conclusions.

2. Governing equations of generalised Galerkin method

The equations of the generalised Galerkin's method for a curved layer of constant curvature and thickness may be obtained straightforwardly from the governing differential equations of motion and the traction-free boundary conditions. However, we derive here these equations from Hamilton principle in order to highlight the variational nature of this method in elastodynamics.

We consider the plane strain state of a layer. Its thickness h is chosen as a length scale, and stresses are scaled by ρc_2^2. Here ρ is the material density, and (c_1, c_2) are the (P, S) wave speeds. The scaled radius of curvature of the centreline of a layer is designated as r_0. The scaled displacement in the circumferential direction (along the θ- axis) is $\tilde{u}(r, \theta, t)$, the scaled displacement in the radial direction (along the r- axis) is $\tilde{v}(r, \theta, t)$. The analysis is restricted to free wave propagation in the absence of external forces.

The kinetic energy is:

$$T = \frac{1}{2}\rho h^4 \int_{r_0-1/2}^{r_0+1/2} \int_{\theta_1}^{\theta_2} \left[\left(\frac{\partial \tilde{u}(r,\theta,t)}{\partial t} \right)^2 + \left(\frac{\partial \tilde{v}(r,\theta,t)}{\partial t} \right)^2 \right] r dr d\theta \tag{1}$$

The potential energy is (here $\alpha = \frac{c_1}{c_2}$):

$$V = \frac{1}{2}\rho h^2 c_2^2 \int_{r_0-1/2}^{r_0+1/2} \int_{\theta_1}^{\theta_2} \left[\left(\alpha^2 \frac{\partial \tilde{u}(r,\theta,t)}{\partial r} + \left(\alpha^2 - 2\right) \frac{1}{r} \left(\frac{\partial \tilde{v}(r,\theta,t)}{\partial \theta} + \tilde{u}(r,\theta,t) \right) \right) \frac{\partial \tilde{u}(r,\theta,t)}{\partial r} + \left(\alpha^2 \frac{1}{r} \left(\frac{\partial \tilde{v}(r,\theta,t)}{\partial \theta} + \tilde{u}(r,\theta,t) \right) \right. \right.$$
$$\left. \left. + \left(\alpha^2 - 2\right) \frac{\partial \tilde{u}(r,\theta,t)}{\partial r} \right) \frac{1}{r} \left(\frac{\partial \tilde{v}(r,\theta,t)}{\partial \theta} + \tilde{u}(r,\theta,t) \right) + 2 \left(r \frac{\partial}{\partial r} \left(\frac{\tilde{v}(r,\theta,t)}{r} \right) + \frac{1}{r} \frac{\partial \tilde{u}(r,\theta,t)}{\partial \theta} \right)^2 \right] r dr d\theta \tag{2}$$

These formulas are substituted in the action integral $H = \int_{t_1}^{t_2} [T - V] dt$, variation $\delta H = 0$ is taken and standard by-parts integration is performed. Then time dependence is taken as $\exp(-\mathrm{i}\omega t)$, i.e. $\tilde{u}(r, \theta, t) = u(r, \theta)\exp(-\mathrm{i}\omega t)$, $\tilde{v}(r, \theta, t) = v(r, \theta)\exp(-\mathrm{i}\omega t)$, this multiplier is omitted and the frequency parameter is introduced as $\Omega = \frac{\omega h}{c_1}$. The system of two variational equations is obtained by equating to zero expressions containing the independent variations $\delta u(r, \theta)$ and $\delta v(r, \theta)$:

$$\int_{r_0-1/2}^{r_0+1/2} \int_{\theta_1}^{\theta_2} \left[\Omega^2 u(r,\theta) + \frac{\partial \sigma_{rr}(r,\theta)}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}(r,\theta)}{\partial \theta} + \frac{1}{r} \left(\sigma_{rr}(r,\theta) - \sigma_{\theta\theta}(r,\theta) \right) \right] \delta u(r,\theta) r dr d\theta - \int_{\theta_1}^{\theta_2} \sigma_{rr}(r,\theta) \delta u(r,\theta) r \Big|_{r=r_0-1/2}^{r=r_0+1/2} d\theta = 0 \tag{3a}$$

$$\int_{r_0-1/2}^{r_0+1/2} \int_{\theta_1}^{\theta_2} \left[\Omega^2 v(r,\theta) + \frac{\partial \sigma_{r\theta}(r,\theta)}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta\theta}(r,\theta)}{\partial \theta} + \frac{2\sigma_{r\theta}(r,\theta)}{r} \right] \delta v(r,\theta) r dr d\theta - \int_{\theta_1}^{\theta_2} \sigma_{r\theta}(r,\theta) \delta v(r,\theta) r \Big|_{r=r_0-1/2}^{r=r_0+1/2} d\theta = 0 \tag{3b}$$

Non-dimensional stresses in Eq (3) are: