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Reply to: Comments on “nonlinear vibration of viscoelastic beams, described using fractional order derivatives”

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The authors wish to thank Professor R.M. Lin for his interest and comments.In the opinion presented in paper [1], the formulae are said to be mathematically incorrect and it is suggested that their application to the problem considered in paper [2] may lead to serious modelling errors. The symbol ${}_{-\infty}D_t^{\alpha}[\bullet]$ denotes the fractional derivative of the order α with respect to t when the lower limit of the integral in the definition of the fractional derivative is $a = -\infty$.

$$ {}_{-\infty}D_t^{\alpha}[\sin \lambda t] = \lambda^{\alpha} \sin(\lambda t + \alpha\pi/2) = \lambda^{\alpha}[\sin(\lambda t)\cos(\alpha\pi/2) + \cos(\lambda t)\sin(\alpha\pi/2)], \tag{1} $$

$$ {}_{-\infty}D_t^{\alpha}[\cos \lambda t] = \lambda^{\alpha} \cos(\lambda t + \alpha\pi/2) = \lambda^{\alpha}[-\sin(\lambda t)\sin(\alpha\pi/2) + \cos(\lambda t)\cos(\alpha\pi/2)], \tag{2} $$

First of all, it must be underlined that only a steady state vibration of viscoelastic beams is considered in our paper [2]. The analysis of transient vibration is out of the scope of the paper. The existence of the steady state vibration of typical systems with damping term, described using fractional derivatives, has also been numerically proved in paper [1].

In paper [2] we also emphasized that formulae (1) and (2) are valid only in the case when the lower limit in the definition of the fractional derivative is moved to $a = -\infty$. In this case, formulae (1) and (2) are valid for both of the considered definitions of fractional derivatives i.e., those given by Riemann-Liouville and Caputo. Therefore, it is not important to provide a precise statement, comprising a definition of the above-mentioned fractional derivatives.

Formulae (1) and (2) can be found in a well-known monograph [3, Page 311], assuming that the lower limit in the Riemann-Liouville definition of fractional derivative is moved to the minus infinity. Moreover, in paper [4] and in the context of nonlinear dynamics, the considered formulae are strictly derived assuming that time approaches infinity where the Caputo

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definition of fractional derivative is used only. This means that formulae (1) and (2) are valid in the asymptotic sense for both of the above-mentioned definitions of fractional derivatives.

In paper [1], the author found numerically some significant differences between his numerical results and formulae (1) and (2). For the sine function, these differences are significant only in the initial range of time, equalling approximately two periods of the function. These results are independent of the order of the fractional derivative. We made our own calculation using Maple's *fracdiff* function and obtained very similar results. In conclusion, numerical results are in agreement with the results based on the analytical formula (1) for t greater than a few periods of the sine function.

However, our numerical results, presented in Figs. 1 and 2, are not in agreement with that presented in Ref. [1] for the cosine function. In our opinion, the results presented in Fig. 3a (in paper [1]) are probably true for $q = 0.6$ (not for $q = 0.2$, as shown) whereas Fig. 3b shows the results for $q = 0.2$ ($\alpha = 0.2$ in our notation). After these changes, the results presented in this note and in paper [1] are in agreement.

Moreover, Figs. 3 and 4 show what happens when time t is far from $t = 0$. In these figures, the results from the analytical formulae are shown as the solid lines, the values of the Riemann-Liouville derivatives are presented as the dashed dot lines whereas those of the Caputo derivatives are shown as the dotted lines. It is apparent that, for $\alpha = 0.6$, numerical and analytical results are approximately equal for $t > 4\pi$, i.e., very fast, and analytical formula (2) is a good approximation of both the Riemann-Liouville and the Caputo fractional derivatives for time values far from zero. For $\alpha = 0.2$, some significant differences between the values of the Caputo derivative, the Riemann-Liouville derivative and those obtained from formula (2) are observed, even for $t > 100\pi$. This means that the asymptotic values of the Caputo derivative of the cosine function and those of the Riemann-Liouville derivative of the cosine function are substantially different for α taken from the range $(0, \alpha_{cr})$, where α_{cr} is currently unknown and should be specified.

In order to clarify how high the value of α_{cr} is, let us compare relative differences between the peak values of the Caputo derivatives of the cosine function obtained numerically with the peak values resulting from the analytical formula (2) and/or the peak values of the Riemann-Liouville derivatives obtained numerically. The relationship between the Riemann-Liouville and the Caputo derivatives is (see Ref. [5], Eq. (14)):

$$ {}^{RL}_{a}\mathrm{D}^{\alpha}_{t}[y(t)] = {}^{C}_{a}\mathrm{D}^{\alpha}_{t}[y(t)] + \frac{y(a)}{(t-a)^{\alpha}\Gamma(1-\alpha)}, \tag{3} $$

where the symbols ${}^{RL}_{a}\mathrm{D}^{\alpha}_{t}[y(t)]$ and ${}^{C}_{a}\mathrm{D}^{\alpha}_{t}[y(t)]$ denote the Riemann-Liouville and the Caputo derivatives with the lower limit at a and with respect to time t, respectively. Moreover, Γ is the gamma function. Please note that the relationship (3) is valid for all $t > 0$.

It is obvious from Eq. (3) that, for $y(t) = \sin \lambda t$ and $a = 0$, $y(0) = 0$ and both of the discussed derivatives are equal for all t; this is in agreement with the numerical results presented in Ref. [1] and obtained by the authors using Maple. However, for $y(t) = \cos \lambda t$ and $a = 0$, $y(0) = 1$ and there are differences between the values of both derivatives. Moreover, the numerical results show that the Riemann-Liouville derivative from the cosine function is an approximately periodic function and the analytical formula is in agreement with the numerical results. In the context discussed here, only the differences for large values of t are important because the asymptotic behavior of the derivatives of the cosine function is of prime interest. If $a = 0$, from (3), the difference between the discussed derivatives is given by the following function of time:

$$ \tilde{\Delta}(t) = \frac{1}{t^{\alpha}\Gamma(1-\alpha)}, \tag{4} $$

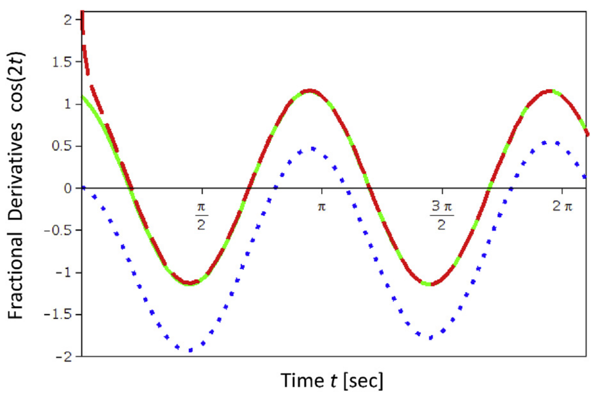

Fig. 1. Fractional derivatives of cos (2t) with $\alpha = 0.2$. —— Formula (1), - - - - Riemann-Liouville derivative, ······ Caputo derivative.