Journal of Sound and Vibration xxx (2018) 1–10

Contents lists available at ScienceDirect

Journal of Sound and Vibration

journal homepage: www.elsevier.com/locate/jsvi

Second-order moment of the first passage time of a quasi-Hamiltonian oscillator with stochastic parametric and forcing excitations

H. Vanvinckenroye [a,b], V. Denoël [a,*]

[a] Structural Engineering Division, Faculty of Applied Sciences, University of Liège, Liège, Belgium
[b] FRIA (F.R.S.-F.N.R.S), National Fund for Scientific Research, Brussels, Belgium

ARTICLE INFO

Article history:
Received 4 August 2017
Revised 1 March 2018
Accepted 2 March 2018
Available online XXX

Keywords:
Stochastic stability
Parametric oscillator
First passage time
Mathieu equation
Generalized Pontryagin equation

ABSTRACT

This work focuses on the stochastic version of the linear Mathieu oscillator with both forced and parametric excitations of small intensity. In this *quasi-Hamiltonian* oscillator, the concept of energy stored in the oscillator plays a central role and is studied through the first passage time, which is the time required for the system to evolve from a given initial energy to a target energy. This time is a random variable due to the stochastic nature of the loading. The average first passage time has already been studied for this class of oscillator. However, the spread has only been studied under purely parametric excitation. Extending to combinations of both forcing and parametric excitations, this work provides a closed-form solution and a thorough analytical study of the coefficient of variation of the first passage time of the energy in this system. Simple asymptotic solutions are also derived in some particular ranges of parameters corresponding to different regimes.

1. Introduction

This paper is concerned with the stochastic version of the undamped Mathieu oscillator, governed by equation

$$\ddot{x}(t) + [1 + u(t)]\,x(t) = w(t), \tag{1}$$

subject to the forced excitation $w(t)$ and to the parametric excitation $u(t)$, and where $x(t)$ is the state variable as a function of time t. As an example, a vertical motion of the support of a pendulum in the gravity field generates this kind of parametric excitation while a horizontal motion generates a forcing excitation [1]. As another example, the deflection of a cable subjected to an axial oscillation of one anchorage is described by a similar Mathieu equation [2]. The rotative equilibrium of tower cranes under gusty wind can also be written in a similar format [3,4].

The current work further assumes that this stochastic oscillator is submitted to small forced and parametric excitations which owes it to be classified as a *quasi-Hamiltonian* oscillator. For this class of oscillators, the concept of total internal energy plays a central role. It finds applications in wave energy harvesting [5–8], capsizing and rolling motions of ships under stochastic wave excitation [9,10] and several other biological applications such as protein folding [11]. Using the appropriate non-dimensionalization and discarding the nonlinear governing components, the governing equations of a large number of applications can be cast under the format of Equation (1) where $u(t)$ and $w(t)$ are stochastic processes.

* Corresponding author.
E-mail address: v.denoel@uliege.be (V. Denoël).

https://doi.org/10.1016/j.jsv.2018.03.001
0022-460X/

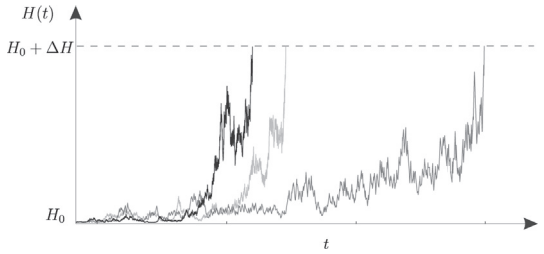

Fig. 1. Three time evolutions of the energy H of a stochastic oscillator from $H_0 = 10^{-5}$ to $H = 10^{-2}$ with white noise excitations of intensities $S_u = 0.01$ and $S_w = 0.5 \times 10^{-5}$.

For the considered governing equation, any large value of the generalized coordinate x is encountered with probability one in the undamped case [12]. For this reason and those that have been mentioned in Ref. [13], we investigate the time required for the system to reach a certain displacement or amplitude, given an initial condition, or to reach a given energy barrier departing from a lower initial total internal energy level. This is known as a *first passage* problem. As the system is stochastically excited, the first passage time is a random variable. Fig. 1 presents three realizations of the energy H of the oscillator departing from a small initial energy H_0 and reaching a larger energy level $H_0 + \Delta H$. Each realization provides a different first passage time.

There are very few problems where the complete statistical distribution of the first passage time is available in closed form [14]. For more complex problems, even such as the problem considered here, perturbation methods or numerical techniques can be used to provide approximations of the exact solution. In particular Monte Carlo simulations, based on realizations as shown in Fig. 1, are known to be versatile and accurate, although highly time consuming. Other approaches are based on the solution of the (generalized) Pontryagin equation(s) [12,15–18], sometimes with the finite difference method [19,20]; alternative methods to determine the transient evolution of joint probability density functions include path integral approaches [21–24], smooth particle hydrodynamics or other Lagrangian methods [25], semi-analytical methods such as the Galerkin projection scheme [26,27], the Poisson distribution based assumptions [28] or other applications of the perturbation method in evolutionary spectral analysis [29].

In this paper, we develop an analytical solution. Analytical methods are usually not able to determine the complete distribution of the first passage time and are limited to its first few statistical moments.

In particular, the mean first passage time provides a first apprehension of the phenomenon so that many stochastic oscillators are first studied by means of their mean first passage times [30–33]. The variance of the first passage time also reflects the range of the possible observed first passage times in real conditions and is therefore interesting in a direct simulation. It also provides a valuable information as to the sample distribution of the mean first passage time, as it depend on the parent distribution of this random variable. With this respect, confidence intervals of observed mean first passage times basically depend on the spread of this random variable.

In this paper, we further restrict the considered problem to cases where $u(t)$ and $w(t)$ are δ-correlated processes. Under this limitation and considering the system to be quasi-Hamiltonian, closed-form solutions exist for the distribution of the first passage time in the undamped configuration ($\xi = 0$) and without external forcing term ($w = 0$) [14,34]. In this latter case, the stochastic differential equation governing the energy is a geometric differential equation. The first passage time of the energy level H_c, starting from a lower initial energy H_0 can be solved explicitly [14] and follows an inverse Gaussian distribution with mean $\frac{4}{S_u} \ln\left(H_c/H_0\right)$ and shape-parameter $\frac{2}{S_u} \ln\left(H_c/H_0\right)^2$. In other or more general cases, the distribution takes very complicated expressions. In this paper, we derive a simple explicit solution for the second-order moment (variance) of the first passage time and provide corresponding solutions in the existing limiting cases, i.e. under forced excitation only or under parametric excitation only. In Section 2, the considered problem is posed. It is solved, validated and discussed in Sections 3 and 4.

2. Problem statement

The undamped, externally and parametrically forced oscillator is governed by the governing equation (1) where $u(t)$ and $w(t)$ are δ-correlated noises of small intensities S_u and S_w, such that $E[u(t)u(s)] = \delta(t-s)S_u$, $E[w(t)w(s)] = \delta(t-s)S_w$ and $E[u(t)w(s)] = \delta(t-s)S_{uw}$.

Since the problem at hand is particularly interesting when the intensities of the excitations are small, the considered oscillator actually happens to be a *quasi-Hamiltonian* system for which the total internal energy (also referred to as the *Hamiltonian*) $H(t)$, defined by

$$H = \frac{x^2}{2} + \frac{\dot{x}^2}{2}, \tag{2}$$

evolves on a slow time scale [15]. Indeed, the energy balance of the governing equation, obtained by time integration of the power fluxes, yields

$$\frac{\dot{x}^2}{2} + \frac{x^2}{2} + \int (u\,x\,\dot{x})\,\mathrm{d}t = \int w\,\dot{x}\,\mathrm{d}t, \tag{3}$$

Please cite this article in press as: H. Vanvinckenroye, V. Denoël, Second-order moment of the first passage time of a quasi-Hamiltonian oscillator with stochastic parametric and forcing excitations, Journal of Sound and Vibration (2018), https://doi.org/10.1016/j.jsv.2018.03.001