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A multi-harmonic generalized energy balance method for studying autonomous oscillations of nonlinear conservative systems

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ABSTRACT

The Harmonic Balance Method (HBM) is a frequency-domain based approximation approach used for obtaining the steady state periodic behavior of forced dynamical systems. Intrinsically these systems are non-autonomous and the method offers many computational advantages over time-domain methods when the fundamental period of oscillation is known (generally fixed as the forcing period itself or a corresponding sub-harmonic if such behavior is expected). In the current study, a modified approach, based on He's Energy Balance Method (EBM), is applied to obtain the periodic solutions of conservative systems. It is shown that by this approach, periodic solutions of conservative systems on iso-energy manifolds in the phase space can be obtained very efficiently. The energy level provides the additional constraint on the HBM formulation, which enables the determination of the period of the solutions.

The method is applied to the linear harmonic oscillator, a couple of nonlinear oscillators, the elastic pendulum and the Henon-Heiles system. The approach is used to trace the bifurcations of the periodic solutions of the last two, being 2 degree-of-freedom systems demonstrating very rich dynamical behavior. In the process, the advantages offered by the current formulation of the energy balance is brought out. A harmonic perturbation approach is used to evaluate the stability of the solutions for the bifurcation diagram.

1. Introduction

The harmonic balance is an established methodology for obtaining periodic solutions of forced dynamical systems [1–4]. The method usually requires a priori knowledge of the fundamental frequency of oscillation. The differential equations of motion are transformed to nonlinear algebraic equations using a Galerkin weighted residual approach into the frequency domain. The approximation representing the solution is in the form of a trigonometric series and the unknowns are the harmonic coefficients. Since the Harmonic Balance Method (HBM) is a well established technique in non-autonomous systems, the method is not described in detail in this paper and the interested reader can refer any of the literature listed above.

Dynamical systems with no external time-varying excitations are known as autonomous dynamical systems. The systems among these for which a conserved total energy is defined are known as conservative dynamical systems. The phase-space motion of these systems is constrained to a $2N-1$ dimensional manifold in the $2N$ dimensional phase-space. The total energy

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of the system essentially acts as a constraining factor in the dynamics of these systems [5]. Thus the phase-space trajectory of conservative single degree-of-freedom (SDOF) systems may be expressed using only one independent variable at each energy level. As a consequence, no SDOF system is capable of showing irregular behavior. For higher degrees of freedom systems with the total energy serving as the only invariant, irregular motion is possible and such systems have been very well studied.

In his paper [6], He proposed a very simple energy balance approach for studying autonomous nonlinear oscillations which has now been extensively studied and expanded into He's variational method [7,8]. Many studies have revealed that He's variational method, being an approximate analytic method, is very efficient for obtaining frequency-amplitude relations for nonlinear oscillators [9,10]. The current paper proposes a hybrid Harmonic-Energy Balance Method (HEBM) seeking to address some issues that He's balance raises.

Existing literature on energy balance methods explore the solution for displacement-only initial conditions. This has been possible since largely (save a few exceptions), only SDOF systems have been studied. The current work expands on this pretext and provides a more generic formulation of energy balance through HBM applicable to higher dimensional systems. As will be brought out, the current formulation in addressing some restrictive features of He's method, may be considered as a super-set of existing implementations.

The study uses the Galerkin weighted residual method for casting the system, thus providing the formulation for a general HBM and shows that bifurcation studies may be conducted on any system by sweeping over the parameter(s) of interest. Moreover, since HBM uses an iterative solver, the current work establishes the relation between the initial guess provided to the solver and the initial condition the converged solution would corresponds to. Although some studies [11] have shown that such a method may be extended even to systems with discontinuities using a modified Lindstedt-Poincare method, the current work will focus exclusively on smooth conservative systems.

2. HEBM formulation

A general nonlinear conservative dynamical system with N degrees-of-freedom may be expressed as,

$$\begin{aligned} \mathbf{M}\ddot{\mathbf{X}} + \mathbf{K}\mathbf{X} + \mathbf{F_{nl}}(\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}}, \dots) &= 0 \\ H(\mathbf{X}, \dot{\mathbf{X}}, \ddot{\mathbf{X}}, \dots) &= E \end{aligned} \tag{1}$$

with $\mathbf{X}$ representing an $N \times 1$ vector of displacements and $\mathbf{M}$ and $\mathbf{K}$ representing $N \times N$ coefficient matrices. $\mathbf{F_{nl}}$ represents an arbitrary nonlinear function of displacement and velocity and H represents the Hamiltonian function of the system. It must be noted that a viscous dissipation term of the form $\mathbf{C}\dot{X}$ is omitted here since the development is specific to conservative systems, where a Hamiltonian may be defined.

In order to make the unknown frequency of the periodic response appear explicitly in the equations, a non-dimensional time parameter $\tau = \omega t$, is introduced; here ω is not known aprioiri. Denoting differentiations with respect to the new time scale with primes, the equations of motion reduce to,

$$\omega^2 \mathbf{M}\mathbf{X}'' + \mathbf{K}\mathbf{X} + \mathbf{F_{nl}}(\mathbf{X}, \omega\mathbf{X}', \omega^2\mathbf{X}'', \dots) = 0 \tag{2}$$

In HBM [1] the vector $\mathbf{X}$, $\begin{bmatrix} x_1 & x_2 & \dots & x_N \end{bmatrix}^T$ is approximated by $\widehat{\mathbf{X}}$, a Fourier series truncated to N_h harmonics as shown below,

$$\widehat{\mathbf{X}} = \mathbf{A_0} + \Sigma_{i=1}^{N_h} \mathbf{A_i} \cos(i\tau) + \mathbf{B_i} \sin(i\tau) \tag{3}$$

with $\mathbf{A_0}$, $\mathbf{A_i}$ and $\mathbf{B_i}$ being $N \times 1$ vectors of corresponding Fourier coefficients. This system is cast in a Galerkin form with a $N(2N_h + 1) \times 1$ vector of unknowns $\widetilde{\mathbf{X}}$ as,

$$\widetilde{\mathbf{X}} = \{\mathbf{A_0^T}\ ,\ \mathbf{A_1^T}\ ,\ \mathbf{B_1^T}\ \dots,\ \mathbf{A_N^T}\ ,\ \mathbf{B_N^T}\}^{\mathbf{T}}$$

The residual vector is defined by writing the system in terms of $\widehat{\mathbf{X}}$.

$$\mathbf{R} = \mathbf{M}\omega^2\widehat{\mathbf{X}}'' + \mathbf{K}\omega\widehat{\mathbf{X}} + \widehat{\mathbf{F_{nl}}}$$

with $F(\widehat{X})$ representing a truncated form of the nonlinear terms in eq. (2). Minimizing $R(\widehat{X})$ over a period $\tau \in (0, 2\pi)$ by using eq. (3) leads to [4],

$$\begin{aligned} &\int_0^{2\pi} \mathbf{R}(\mathbf{X}) \frac{\partial \widehat{\mathbf{X}}}{\partial \mathbf{A_0}} d\tau + \sum_{i=1}^{N_h} \left[\int_0^{2\pi} \mathbf{R}(\mathbf{X}) \frac{\partial \widehat{\mathbf{X}}}{\partial \mathbf{A_i}} d\tau + \int_0^{2\pi} \mathbf{R}(\mathbf{X}) \frac{\partial \widehat{\mathbf{X}}}{\partial \mathbf{B_i}} d\tau \right] = 0 \\ \implies &\int_0^{2\pi} \mathbf{R}(\mathbf{X}) d\tau + \sum_{i=1}^{N_h} \left[\int_0^{2\pi} \mathbf{R}(\mathbf{X}) \cos(i\tau) d\tau + \int_0^{2\pi} \mathbf{R}(\mathbf{X}) \sin(i\tau) d\tau \right] = 0 \end{aligned} \tag{4}$$