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Free vibration of functionally graded beams and frameworks using the dynamic stiffness method

J.R. Banerjee[*], A. Ananthapuvirajah

School of Mathematics, Computer Science and Engineering, City, University of London, Northampton Square, London EC1V 0HB, UK

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A B S T R A C T

The free vibration analysis of functionally graded beams (FGBs) and frameworks containing FGBs is carried out by applying the dynamic stiffness method and deriving the elements of the dynamic stiffness matrix in explicit algebraic form. The usually adopted rule that the material properties of the FGB vary continuously through the thickness according to a power law forms the fundamental basis of the governing differential equations of motion in free vibration. The differential equations are solved in closed analytical form when the free vibratory motion is harmonic. The dynamic stiffness matrix is then formulated by relating the amplitudes of forces to those of the displacements at the two ends of the beam. Next, the explicit algebraic expressions for the dynamic stiffness elements are derived with the help of symbolic computation. Finally the Wittrick-Williams algorithm is applied as solution technique to solve the free vibration problems of FGBs with uniform cross-section, stepped FGBs and frameworks consisting of FGBs. Some numerical results are validated against published results, but in the absence of published results for frameworks containing FGBs, consistency checks on the reliability of results are performed. The paper closes with discussion of results and conclusions.

1. Introduction

In recent years interest in functionally graded material (FGM) has grown enormously. The progress made in understanding this material has been phenomenal. One great advantage of FGM is that the properties vary gradually in a continuous manner within the material so that there is no abrupt change or mismatch of the properties which can cause delamination or other problems generally associated with fibre-reinforced composites. Thus, FGM can be designed in a way to have the properties of ceramic at one end and those of metal at the other so that the thermal resistance of ceramic and the mechanical behaviour of metal can be exploited to advantage to guarantee structural integrity. Consequent on this, researchers have been continually motivated to use various techniques and methodologies to deal with this exciting material in order to enhance its state-of-the-art. There are now excellent books [1–4] available on the subject. As potential application of FGM, beams which are extensively used in civil, mechanical, aeronautical and other branches of engineering as principal load carrying structural members can be investigated for their free vibration characteristics. Investigators have expended considerable efforts which have led to the insurgence of massive literature on the free vibration behaviour of Functionally Graded Beams (FGBs). A number of theories and methodologies have been proposed to study the free vibration characteristics of FGBs. Foremost

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[*] Corresponding author.
E-mail address: j.r.banerjee@city.ac.uk (J.R. Banerjee).

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amongst these are the applications of direct analytical procedure using the governing differential equations of motion [5–19], finite element [20–22], Rayleigh-Ritz [23], finite volume [24–26], differential quadrature [27], differential transformation [27,28] and transfer function [29,30] methods. Recently the dynamic stiffness method (DSM) has also been proposed [31,32]. The current paper stems from the previously published DSM theories. The entire formulation using DSM in this paper is accomplished in the real domain as opposed to previous formulations which used complex arithmetic when developing the element dynamic stiffness matrices [31,32]. Another important further development reported in this paper is the derivation of explicit algebraic expressions for the dynamic stiffness elements using symbolic computation [33–35]. The explicit expressions for the dynamic stiffness elements are particularly useful in optimisation studies and also when some, but not all of the stiffnesses are needed. Of particular significance of this investigation is the application of DSM to analyse the free vibration characteristics of stepped FGBs and frameworks containing FGBs. The substantial advantages of the DSM over the conventional finite element method (FEM) are well known [36–38]. The DSM is often called an exact method because in sharp contrast to chosen approximate shape function assumed in the FEM, the DSM uses exact shape function obtained from the analytical solution of the governing differential equation of motion of the structural element in free vibration. The uncompromising accuracy of the DSM in all frequency ranges and its independency on the number of elements used in the analysis make it particularly appealing in free vibration analysis. Within this pretext, the application of the DSM in the free vibration analysis of FGBs and frameworks containing FGBs is considered to be a welcome development. The solution technique used in the DSM is robust, particularly when the well-established algorithm of Wittrick and Williams [39], known as Wittrick-Williams algorithm in the literature, is applied. The algorithm ensures that no natural frequency of the structure is missed, and it has featured in literally hundreds of papers. It is worth noting that earlier investigations on the free vibration of FGBs were focused on individual FGBs except for a few isolated cases where stepped FGBs with collinear axes were reported [40,41]. Accordingly, the literature on the free vibration of frameworks containing FGBs is virtually non-existent. One of the essential purposes of this paper is to fill this gap.

2. Theory

In a right-handed Cartesian coordinate system, Fig. 1 shows a uniform rectangular cross-section FGB of length L, width b and thickness h. The beam material has Young's Modulus E and mass density ρ which can vary through the thickness direction (Z) of the cross-section according to the following power law distribution [14,17,30,32]:

$$E(z) = (E_t - E_b)\left(\frac{z}{h} + \frac{1}{2}\right)^k + E_b, \quad \rho(z) = (\rho_t - \rho_b)\left(\frac{z}{h} + \frac{1}{2}\right)^k + \rho_b. \tag{1}$$

where E_t and E_b are the Young's moduli, and ρ_t and ρ_b are the densities at the top and bottom surfaces of the beam, respectively.

In Eq. (1), k ($k \geq 0$) is the power law index parameter which indicates the material property variation through the beam thickness. The parameter k has been extensively discussed in the literature [14–19] and hence it is not elaborated here. However, three special cases maybe observed. Clearly $k = 1$ indicates a linear variation of the composition between the top and bottom surfaces of the beam, $k = 0$ represents the case when the beam is made of full material of the top surface whereas infinite k represents the case when the beam is made of full material of the bottom surface.

2.1. Governing differential equations of motion and solution

The classical Bernoulli-Euler theory is considered here so that the effects of shear deformation and rotary inertia that are relevant to the Timoshenko beam theory are assumed to be small and hence disregarded in the analysis. Referring to Fig. 1, the displacements u_1, v_1 and w_1 along the X, Y and Z directions of a point on the cross-section are given by Refs. [6,30,32]:

$$u_1 = 0 \quad v_1(y,z,t) = v(y,t) - z\frac{\partial w(y,t)}{\partial y}, \quad w_1(y,z,t) = w(y,t) \tag{2}$$

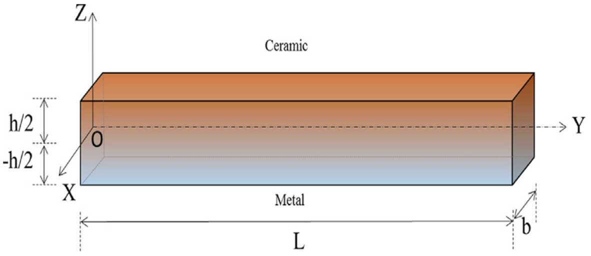

Fig. 1. Coordinate System and dimensions of a functionally graded beam.