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Automated computation of autonomous spectral submanifolds for nonlinear modal analysis

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ABSTRACT

We discuss an automated computational methodology for computing two-dimensional spectral submanifolds (SSMs) in autonomous nonlinear mechanical systems of arbitrary degrees of freedom. In our algorithm, SSMs, the smoothest nonlinear continuations of modal subspaces of the linearized system, are constructed up to arbitrary orders of accuracy, using the parameterization method. An advantage of this approach is that the construction of the SSMs does not break down when the SSM folds over its underlying spectral subspace. A further advantage is an automated a posteriori error estimation feature that enables a systematic increase in the orders of the SSM computation until the required accuracy is reached. We find that the present algorithm provides a major speed-up, relative to numerical continuation methods, in the computation of backbone curves, especially in higher-dimensional problems. We illustrate the accuracy and speed of the automated SSM algorithm on lower- and higher-dimensional mechanical systems.

1. Introduction

A fundamental notion in decomposing nonlinear mechanical oscillations, is the *nonlinear normal mode* (NNM) concept of Rosenberg [1], who defined a nonlinear normal mode as a synchronous periodic oscillation that reaches its maximum in all modal coordinates at the same time. An alternative definition of a NNM, proposed by Shaw and Pierre [2], is an invariant manifold that serves as the nonlinear continuation of two-dimensional subspaces formed by normal modes of the linearized system. Shaw and Pierre seek such invariant manifolds as graphs over those two-dimensional subspaces. For several extensive discussions about these two NNM definitions, we refer the reader to the work of Kerschen et al. [3], Peeters et al. [4], Mikhlin and Avramov [5] and Vakakis et al. [6].

If one relaxes the synchronicity requirement of Rosenberg, a clear relationship between the above two views on NNMs emerges for conservative oscillatory systems by the Lyapunov subcenter-manifold Theorem [7,8]. Indeed, under appropriate non-resonance conditions, these references guarantee the existence of a unique, analytic and robust Shaw—Pierre-type invariant manifold tangent to each two-dimensional modal subspace of the linearized system. This manifold, in turn, is filled with Rosenberg-type periodic orbits.

In a non-conservative setting, this geometrical relationship between the two classic NNM concepts no longer holds, as periodic orbits become rare and isolated in the phase space, whereas infinitely many invariant manifolds tangent to each two-dimensional modal subspace will exist. A unified approach has been proposed by Haller and Ponsioen [9] to clarify the relationship between the Rosenberg and Shaw—Pierre NNM concepts. Specifically [9], defines a nonlinear normal mode simply as

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a recurrent motion with finitely many frequencies. Included in this theory is a trivial NNM or fixed point (no frequencies), a periodic NNM (the frequencies are rationally commensurate, as for a Rosenberg-type periodic orbit) and a quasiperiodic NNM (the frequencies are rationally incommensurate, with the orbit filling an invariant torus).

Using this NNM definition, Haller and Ponsioen [9] define a spectral submanifold (SSM) as the smoothest invariant manifold tangent to a modal subspace of a NNM. They then invoke rigorous existence, uniqueness and persistence results for autonomous and non-autonomous SSMs, providing an exact mathematical foundation for constructing nonlinear reduced-order models over appropriately chosen spectral subspaces. These models are obtained by reducing the full dynamics to the exactly invariant SSM surfaces, tangent to those subspaces.

More recently, Szalai et al. [10] have shown that the dynamics on a single-mode SSM can be seen as a nonlinear extension of the linear dynamics of the underlying modal subspace, making it possible to extract the *backbone curve*, defined as a graph plotting the instantaneous amplitude of vibration as a function of the instantaneous frequency of vibration. This approach to backbone-curve computations assumes that the Lyapunov subcenter-manifold perturbs smoothly to a unique SSM under appropriate non-resonance conditions and under small enough damping, which is consistent with the numerical observations as shown by Kerschen et al. [11], Peeters et al. [12] and Szalai et al. [10].

Computing invariant manifolds tangent to modal subspaces in realistic applications has been a challenge. Prior approaches have mostly focussed on solving the invariance equations that such manifolds have to satisfy (Blanc et al. [13], Pesheck et al. [14] and Renson et al. [15]). These invariance equations have infinitely many solutions, out of which the numerical approaches employed by different authors selected one particular solution. In contrast, the SSM theory employed here guarantees a unique solution that can be approximated with arbitrary high precision via the parameterization method of Cabré et al. [16–18]. In the present work, we describe an automated computational algorithm for two-dimensional SSMs constructed over two-dimensional modal subspaces. This algorithm[1] allows us to compute the SSMs, their reduced dynamics and associated backbone curves to arbitrary orders of accuracy, limited only by available memory. An important feature of the algorithm is a direct a posteriori estimation of the error in computing the SSM at a given approximation order. This error estimate measures directly the extend to which the SSM is invariant. If the error is unsatisfactory, the user can select higher order approximations until the error falls below a required bound.

In technical terms, we construct the SSMs as embeddings of the modal subspaces into the phase space of the mechanical system, as required by the parameterization method (Cabré et al. [16–18]). A major advantage compared to most earlier calculations (Haller and Ponsioen [9]) is that the parameterized construction of SSMs does not break down when the SSM folds over the underlying modal subspace. Another advantage of the method is its suitability for algorithmic implementations for arbitrary orders of accuracy in arbitrary dimensions. For applications of the parameterization method to other types of dynamical systems, we refer the reader to the work of Haro et al. [19], van den Berg and Mireles James [20] and Mireles James [21].

2. System set-up

We consider n-degree-of-freedom, autonomous mechanical systems of the form

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} + \mathbf{f}(\mathbf{y}, \dot{\mathbf{y}}) = \mathbf{0}, \qquad \mathbf{f}(\mathbf{y}, \dot{\mathbf{y}}) = \mathcal{O}\left(|\mathbf{y}|^2, |\mathbf{y}|\,|\dot{\mathbf{y}}|\,, |\dot{\mathbf{y}}|^2\right), \tag{1}$$

where $\mathbf{y} \in \mathbb{R}^n$ is the generalized position vector; $\mathbf{M} = \mathbf{M}^T \in \mathbb{R}^{n\times n}$ is the positive definite mass matrix; $\mathbf{C} = \mathbf{C}^T \in \mathbb{R}^{n\times n}$ is the damping matrix; $\mathbf{K} = \mathbf{K}^T \in \mathbb{R}^{n\times n}$ is the stiffness matrix and $\mathbf{f}(\mathbf{x}, \dot{\mathbf{x}})$ denotes all the nonlinear terms in the system. These nonlinearities are assumed to be of class C^r in $(\mathbf{x}, \dot{\mathbf{x}})$, with $r \in \mathbb{N}^+ \cup \{\infty, a\}$. Here $r \in \mathbb{N}^+$ refers to finite differentiability, $r = \infty$ refers to infinite differentiability, and $r = a$ refer to analyticity, all three of which are allowed in our treatment.

System (1) can be transformed into a set of $2n$ first-order ordinary differential equations by introducing a change of variables $\mathbf{x}_1 = \mathbf{y}, \mathbf{x}_2 = \dot{\mathbf{y}}$, with $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^{2n}$, which gives,

$$\dot{\mathbf{x}} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{pmatrix}\mathbf{x} + \begin{pmatrix} \mathbf{0} \\ -\mathbf{M}^{-1}\mathbf{f}(\mathbf{x}_1, \mathbf{x}_2) \end{pmatrix} = \mathbf{A}\mathbf{x} + \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^{2n}, \quad \mathbf{F}(\mathbf{x}) = \mathcal{O}\left(|\mathbf{x}|^2\right). \tag{2}$$

The transformed system (2) has a fixed point at $\mathbf{x} = \mathbf{0}$, $\mathbf{A} \in \mathbb{R}^{2n\times 2n}$ is a constant matrix and $\mathbf{F}(\mathbf{x})$ is a class C^r function containing all the nonlinearities. Note that the inverse of the mass matrix is well-defined because $\mathbf{M}$ is assumed positive definite.

The linearized part of (2) is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{3}$$

where the matrix $\mathbf{A}$ has $2n$ eigenvalues $\lambda_k \in \mathbb{C}$ for $k = 1,\ldots,2n$. Counting multiplicities, we sort these eigenvalues based on their real parts in the decreasing order,

$$\mathrm{Re}(\lambda_{2n}) \leq \mathrm{Re}(\lambda_{2n-1}) \leq \cdots \leq \mathrm{Re}(\lambda_1) < 0, \tag{4}$$

assuming that the real part of each eigenvalue is less than zero and hence the fixed point is asymptotically stable. We further assume that the constant matrix $\mathbf{A}$ is semisimple, which implies that the algebraic multiplicity of each λ_k is equal to its

[1] SSMtool is available at: http://www.georgehaller.com.