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# Normalized modes at selected points without normalization

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As every textbook on linear algebra demonstrates, the eigenvectors for the general eigenvalue problem $|\mathbf{K} - \lambda\mathbf{M}| = 0$ involving two real, symmetric, positive definite matrices $\mathbf{K}, \mathbf{M}$ satisfy some well-defined orthogonality conditions. Equally well-known is the fact that those eigenvectors can be normalized so that their modal mass $\mu = \boldsymbol{\phi}^T\mathbf{M}\boldsymbol{\phi}$ is unity: it suffices to divide each unscaled mode by the square root of the modal mass. Thus, the normalization is the result of an explicit calculation applied to the modes *after* they were obtained by some means. However, we show herein that the normalized modes are not merely convenient forms of scaling, but that they are actually intrinsic properties of the pair of matrices $\mathbf{K}, \mathbf{M}$, that is, the matrices already "know" about normalization even *before* the modes have been obtained. This means that we can obtain individual components of the normalized modes directly from the eigenvalue problem, and without needing to obtain either all of the modes or for that matter, any one complete mode. These results are achieved by means of the residue theorem of operational calculus, a finding that is rather remarkable inasmuch as the residues themselves do not make use of any orthogonality conditions or normalization in the first place. It appears that this obscure property connecting the general eigenvalue problem of modal analysis with the residue theorem of operational calculus may have been overlooked up until now, but which has in turn interesting theoretical implications.Á



## 1. Introduction

We elaborate in this article on the intimate connection that exists between the modal superposition method familiar to engineers and the method of residues which is typically applied when the governing equations are formulated in the frequency domain and then inverted into the time domain via a Fourier transform. We discover in the process that the residues contain information about the *normalized* modes without ever making use of any orthogonality conditions, either explicitly or implicitly. It should be pointed out that although both the system approach based on Fourier inversion and the modal superposition method are indeed very well known, the connection between these topics is conspicuously absent in books on structural dynamics, wave propagation or linear algebra.

Now, when a structural dynamicist or vibration engineer talks of modes of vibration, s/he refers to patterns of vibration in a structure of finite size which in the absence of external loads can be observed at a countable, discrete set of frequencies. Together with these characteristic frequencies, the modes are typically obtained from the solution of an eigenvalue problem which satisfies some well-known orthogonality conditions. Except for a few degenerate systems, the modes are guaranteed to span the full finite or infinite vector space associated with the active dynamic degrees of freedom, so whether one deals with

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discrete or continuous systems, the modal set is complete in the sense that any arbitrary vector can be expressed in terms of those modes, which allows finding the response to any arbitrary excitation applied anywhere in the system by recourse to modal superposition. The modal weights in such cases are obtained by expressing the load in terms of the modes, which is accomplished with the stated orthogonality conditions. In the absence of damping (attenuation), the modal oscillations, once set up, will continue indefinitely.

On the other hand, a geophysicist dealing with layered half-spaces refers to normal modes as the patterns of waves that can be observed within wave guides, among which the Love, Rayleigh, and Stoneley waves are the best known. Such motion patterns also exist in the absence of external sources, yet they do not remain confined to some local vicinity but forever propagate instead along specific directions. Depending on the dimensionality of the problem, the patterns may decay due to geometric spreading, even when the soil lacks any material attenuation. Unlike the modes in a finite system, a finite number of normal modes may exist over continuous frequency bands and such modes differ only in their characteristic wavelength (or wavenumber). It seems safe to state that the normal modes in geophysics are rarely obtained from the solution to an eigenvalue problem —or even understood as resulting from such a problem— but are more often inferred by means of search techniques that look for singularities in some characteristic functions, such as the well-known Rayleigh function. Just as important, the normal modes are no longer sufficient to fully describe the dynamic response to sources somewhere, although they may dominate the response at large range. Thus, a geophysicist will typically formulate the solution in terms of integral transforms from the frequency-wavenumber domain into the space-time domain, aided in part by contour integrations that take into account the residues at the real poles and supplementing these with so-called branch integrals, the details of which are irrelevant herein. The important point is that within this context, the evaluation of the response via residues is understood as being the normal modes method, even though no eigenvalue problem is explicitly being solved and the concept of orthogonality of modes and the expansion of dynamic sources in terms of such modes weighed by participation factors remains generally alien to the geophysicist.

In this article we elaborate on the intimate connection between the standard modal superposition method and the system approach based on residue theory, both of which are referred to by their practitioners as *the* normal modes method. We rush to add, however, that both of these methods are indeed very well known; the aim herein is not to demonstrate details of these methods that are known and obvious to all, but to demonstrate a property that is rather obscure and has escaped attention so far: That it is possible to compute *normalized* modes at just a few *selected spatial locations* and to do so *without the need to determine the complete set of modes* and without the use of orthogonality conditions.

## 2. Normal modes vs. residues

Consider the eigenvalue problem involving a pair of $N \times N$, real, symmetric matrices $\mathbf{K}, \mathbf{M}$, of which $\mathbf{K}$ is either positive semi-definite or positive definite, and $\mathbf{M}$ is always positive definite:

$$\mathbf{K}\boldsymbol{\phi}_j = \lambda_j \mathbf{M}\boldsymbol{\phi}_j \tag{1}$$

Thus, the eigenvalues are always real and non-negative. This eigenvalue problem satisfies the orthogonality conditions

$$\boldsymbol{\Phi}^T\mathbf{K}\boldsymbol{\Phi} = \mathbb{K} = \mathbb{M}\boldsymbol{\Lambda} = \mathrm{diag}\left(\mu_j\lambda_j\right) = \text{modal stiffness} \tag{2a}$$

$$\boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} = \mathbb{M} = \mathrm{diag}\left(\mu_j\right) = \text{modal mass} \tag{2b}$$

If so desired, the modes can also be normalized so that they attain a unit modal mass, i.e.

$$\boldsymbol{\Psi} = (\mathbb{M})^{-1/2}\boldsymbol{\Phi} \equiv \left\{\boldsymbol{\psi_j}\right\}, \boldsymbol{\psi_j} = \phi_j\Big/\sqrt{\mu_j}, \quad \boldsymbol{\Psi}^T\mathbf{M}\boldsymbol{\Psi} = \mathbf{I} \tag{3}$$

Consider now the matrix pencil

$$\mathbf{Z} = \mathbf{K} - \lambda\mathbf{M} \tag{4}$$

whose inverse is

$$\mathbf{H} = \mathbf{Z}^{-1} = (\mathbf{K} - \lambda\mathbf{M})^{-1} \tag{5}$$

Expressed in terms of the modes, this inverse can be written as

$$\mathbf{H}(\lambda) = \boldsymbol{\Phi}(\mathbb{K} - \lambda\mathbb{M})^{-1}\boldsymbol{\Phi}^T = \left(\boldsymbol{\Phi}\mathbb{M}^{-1/2}\right)(\boldsymbol{\Lambda} - \lambda\mathbf{I})^{-1}\left(\mathbb{M}^{-1/2}\boldsymbol{\Phi}^T\right) = \boldsymbol{\Psi}(\boldsymbol{\Lambda} - \lambda\mathbf{I})^{-1}\boldsymbol{\Psi}^T = -\sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\boldsymbol{\psi_j}\boldsymbol{\psi_j}^T \tag{6}$$

that is, each component of $\mathbf{H}$ is of the form