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Coupling between shear and bending in the analysis of beam problems: Planar case

Ahmed A. Shabana*, Mohil Patel

Department of Mechanical and Industrial Engineering, University of Illinois at Chicago, 842 West Taylor Street, Chicago, IL, 60607, USA

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ABSTRACT

The interpretation of invariants, such as curvatures which uniquely define the bending and twist of space curves and surfaces, is fundamental in the formulation of the beam and plate elastic forces. Accurate representations of curve and surface invariants, which enter into the definition of the strain energy equations, is particularly important in the case of large displacement analysis. This paper discusses this important subject in view of the fact that shear and bending are independent modes of deformation and do not have kinematic coupling; this is despite the fact that kinetic coupling may exist. The paper shows, using simple examples, that shear without bending and bending without shear at an arbitrary point and along a certain direction are scenarios that higher-order finite elements (FE) can represent with a degree of accuracy that depends on the order of interpolation and/or mesh size. The FE representation of these two kinematically uncoupled modes of deformation is evaluated in order to examine the effect of the order of the polynomial interpolation on the accuracy of representing these two independent modes. It is also shown in this paper that not all the curvature vectors contribute to bending deformation. In view of the conclusions drawn from the analysis of simple beam problems, the *material curvature* used in several previous investigations is evaluated both analytically and numerically. The problems associated with the material curvature matrix, obtained using the rotation of the beam cross-section, and the fundamental differences between this material curvature matrix and the Serret-Frenet curvature matrix are discussed.

1. Introduction

The formulation of the continuum elastic forces using a general continuum mechanics approach requires having a complete set of position vector gradients in order to properly define the components of the Green-Lagrange strain tensor or any other general strain measure [1–3]. Having a complete set of position vector gradients requires a full-parameterization of the continuum; two parameters for planar surfaces, and three parameters for volumes. The use of the general continuum mechanics approach in the case of fully-parameterized continuum does not require explicit definition of geometric invariants such as curvature and torsion which uniquely define the geometries of curves and surfaces [4–7].

When gradient deficient continuum models are used, on the other hand, the general continuum mechanics approach cannot be used. In this case, specialized classical beam and plate theories are used to formulate the stress forces [8–12]. In

* Corresponding author.
E-mail addresses: shabana@uic.edu (A.A. Shabana), mpate72@uic.edu (M. Patel).

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these specialized theories, the geometric invariants, such as curvature and torsion, are often used in the formulation of the energy expressions. Accurate definitions of these invariants, therefore, is necessary in order to obtain reliable solutions. A fiber of a beam, for example, can be considered as a space curve that can experience arbitrary bending deformations. The curvature of the fiber is defined as the magnitude of the curvature vector obtained by differentiation of the unit tangent with respect to the arc length. The curvature, often in a simplified form, has been widely used in the literature in the formulation of the bending strain energy [8–12].

In the general theory of continuum mechanics, an infinitesimal volume has six independent modes of deformation; three stretch modes and three shear modes. Simplifying assumptions are used in the development of existing beam theories which are based, for the most part, on one-dimensional parameterization. For example, in most existing beam formulations, the beam cross section is assumed to remain planar and rigid. In the case of shear-deformable planar beams, for example, the shear is defined in terms of the angle between the beam cross section and the normal to the beam centerline. The shear angle is assumed to be totally independent from the beam bending; a beam, at an arbitrary point and along a certain direction, can bend without shear and can shear without bending. This basic assumption, which has been used for decades in the formulation of the beam equations, is also consistent with the general continuum mechanics theory in which pure shear deformation can be achieved independently from all other deformation types. The phrase "shear without bending" will be used in this paper to refer to the case in which shear is the dominant mode of deformation experienced by the continuum. Nonetheless, in some investigations, new curvature definitions that kinematically couple the curvature and shear were introduced [13–15]. In some of these definitions, the angles that define the orientation of the beam cross section are used in the definition of the curvatures. A transformation matrix expressed in terms of these cross section angles is used in a manner that resembles the Serret-Frenet frame transformation used to define the curvature and torsion of a space curve [5,6]. In the Serret-Frenet approach, the tangent, normal, and bi-normal vectors are used to define the Frenet frame which depends only on and can be uniquely defined using one parameter; the curve *arc length*.

When using the first and second fundamental theorems of surfaces [5,6], it is important to recognize that not all curvature vectors are associated with bending deformations. Curvature vectors that involve differentiation twice with respect to the same coordinate line describe the change of the orientation of tangent vectors to fibers and can be associated with bending deformation as will be discussed in this paper. Curvature vectors that result from the differentiation with respect to two different coordinate lines (parameters) can appear in the formulation of the shear as will be demonstrated in this paper.

The absolute nodal coordinate formulation (ANCF) fully-parameterized elements allow for developing new and more general beam models and for investigating the assumptions used in the classical beam and plate theories [16–29]. The use of the position vector gradients as nodal coordinates allows for relaxing the assumptions of small deformation used for most conventional beam and plate elements [30–32]. ANCF position gradient vectors capture accurately arbitrarily large rigid body rotations and high speed spinning motion [33–35]. For this reason, such general elements are suited for evaluating the assumptions used in the classical approaches.

This study, which is concerned with the interpretation of the geometric invariants such as the curvature in the FE large displacement analysis, is motivated by the fact that the definition of curve and surface invariants is fundamental for the accurate beam and plate stress force formulation. The focus in this paper will be on the analysis of planar beams in order to avoid the complexities of the three-dimensional analysis and to be able to obtain simple expressions that can shed light clearly on different geometric definitions. The specific contributions of this paper can be summarized as follows:

1. It is demonstrated analytically, in Section 2, that the position vector gradients can be used to represent the case of shear without bending. To this end, each of the gradient vectors is defined in terms of a stretch coefficient and an angle that defines the orientation of the gradient vector. It is shown that in the case of shear, the transverse gradient vector can have an arbitrary orientation with respect to the normal to the beam fiber. The shear can be non-uniform, while the beam fiber remains straight, demonstrating that bending cannot be defined in terms of the derivative of the shear angle.
2. In Section 3, the accuracy of using the FE approximation in the representation of the shear without bending (SWB) and bending without shear (BWS) is examined. To this end, a fully-parameterized ANCF beam element is used to formulate the beam problem [20]. It is shown that higher-order elements can be used systematically to obtain the SWB and BWS scenarios in the case of arbitrarily large rigid body displacement.
3. It is shown in Section 4 that a curvature vector obtained by differentiation with respect to two different coordinate lines (parameters) may not be associated with bending deformation, and such a curvature vector can appear in the formulation of the SWB problem. This curvature vector can be related to the derivative of the angle that defines the orientation of the cross section with respect to the arc length parameter.
4. Based on the conclusions drawn from the analysis presented in Sections 2 and 3, the definition of the *material curvature* used by other researchers is evaluated in Section 4 [13–15]. Despite the fact that the material curvature is defined in terms of the angle that defines the orientation of the beam cross section, the material curvature was used by some researchers to formulate the beam bending forces.
5. The fundamental differences between the Serret-Frenet approach and the material curvature approach, that employs the cross section orientation matrix, are highlighted in Section 4 of the paper.