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Model-order reduction of lumped parameter systems via fractional calculus

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ABSTRACT

This study investigates the use of fractional order differential models to simulate the dynamic response of non-homogeneous discrete systems and to achieve efficient and accurate model order reduction. The traditional integer order approach to the simulation of non-homogeneous systems dictates the use of numerical solutions and often imposes stringent compromises between accuracy and computational performance. Fractional calculus provides an alternative approach where complex dynamical systems can be modeled with compact fractional equations that not only can still guarantee analytical solutions, but can also enable high levels of order reduction without compromising on accuracy. Different approaches are explored in order to transform the integer order model into a reduced order fractional model able to match the dynamic response of the initial system. Analytical and numerical results show that, under certain conditions, an exact match is possible and the resulting fractional differential models have both a complex and frequency-dependent order of the differential operator. The implications of this type of approach for both model order reduction and model synthesis are discussed.

1. Introduction

Numerical accuracy and computational efficiency have been long-standing challenges in the simulation of dynamical systems. Numerical solutions of complex continuous systems with non-trivial boundary and loading conditions typically require a discretization process to obtain lumped parameter models. The more complex the property spatial distribution (e.g. external loads, material or geometric parameters, boundary conditions), the higher the level of discretization needed to achieve a satisfactory representation of the original continuum system. The increased discretization directly impacts the computational performance and, for large systems, limits the level of achievable accuracy. This issue is even more accentuated when dealing with active control or real-time prediction of the dynamic response of systems under operating conditions for which fast state estimation is a key requirement.

Over the last several decades, these challenges have motivated the rapid growth and development of methodologies for the synthesis of computationally efficient approaches able to reduce the overall size of the models (i.e. of the total number of Degrees Of Freedom - DOF) while maintaining high numerical accuracy and fidelity to the actual dynamics. These techniques, referred to as *model order reduction*, are typically pursued when the dynamic response is sought only at selected locations (the so-called active DOFs) such that the DOFs associated with the remaining locations can be omitted. The reduction procedure

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is not trivial because it must account for the coupling between active and omitted DOFs in order to not change the underlying dynamics of the system.

Many sources in the literature [1–3] provide an extensive review of reduction techniques across a variety of disciplines. In the following, we concentrate only on applications to structural dynamics since this is the emphasis of our study. One widely used technique is Guyan reduction. This approach is also known as static reduction because it does not account for the system inertia and is therefore limited to statics. Reduction techniques for dynamic systems (therefore accounting for the system's inertia) are often based on mode superposition or component mode synthesis [1]. Perhaps one of the most widely used component mode synthesis techniques is the Craig-Bampton method [4,5]. The Craig-Bampton method divides the system into several substructures which are required to be compatible along their shared boundaries. Assuming these boundaries are held fixed, the Craig-Bampton method is able to combine the motion of these boundary points with the displacement modes of the substructures (known as constraint modes). The dynamics of the system can be reduced to a set of both fixed-interface and constraint modes [6].

Many existing order reduction techniques can only provide an approximation of the local response. For instance, the accuracy of the approximation of the Craig-Bampton method is strictly dependent on the number of modes retained in the modal basis. The truncation of the basis should be assessed with respect to either the modal densities associated with the omitted modes or the dynamic content that should be transferred to the active degrees. In a similar way, enlarging the modal basis (i.e. extending the truncation order) comes at the expense of computational performance.

To address these limitations, we explored a reduction order technique based on fractional calculus. We will show that while fractional models contain less DOFs than the original system, often times the dynamic response can be matched exactly at the active degrees. This is an important advantage of our fractional order reduction over reduction order techniques typically used. In addition to the order reduction capabilities, we anticipate that the proposed approach has possible advanced applications to the system identification based only on measured or experimental data. As an example of these capabilities, we will show the application of the fractional models to perform broadband system identification of discrete parameter systems.

While the mathematics of fractional calculus has been extensively studied in the past century, applications are relatively recent. In particular, fractional calculus has seen applications in engineering areas such as vibration control [7,8], visco- and thermo-elasticity [9–11], and wave propagation in complex media [12–14]. The reader is referred to [15–17] for detailed reviews on the fundamentals of fractional calculus. In Appendix A, we define some basic fractional calculus quantities. We anticipate that the methodology discussed below produces fractional differential models of complex and frequency-dependent order. Love [18], Ortigueira [19], Ross [20], and Andriambololona [21] among others have developed the mathematics of complex fractional derivatives as well as potential uses. Adams [22] and Neamaty [23] have developed solutions to certain complex fractional order differential equations. Other authors including Atanackovic [24], Makris [25], and Park [26] have successfully applied complex fractional calculus to viscoelasticity. While the mathematics of complex fractional calculus has been explored and developed, its connection to the actual physical processes being represented can be difficult to grasp. Perhaps Makris has one of the clearest interpretations of a complex order derivative: "… *one may interpret the complex derivative of an arbitrary function as the superposition of complex derivatives of harmonic functions. Evidently, complex-order derivatives modulate the phase and amplitude of harmonic components of a time-dependent function in a more complicated way than real-order derivatives. An important difference between real-valued and complex-valued time derivatives is that phase modulation in the latter case is frequency dependent whereas in the former is not*" [25]. As pointed out here by Makris, complex fractional derivatives can produce functions whose amplitude and phase are both frequency-dependent; this attribute plays a key role in the development of the fractional models. Valerio [27,28] gives a thorough review of complex and variable-order fractional derivatives through the use of Laplace transforms and transfer functions.

The remainder of the paper is structured as follows: in the next section, we present how to obtain an undamped fractional single degree of freedom model having the same dynamic response of an integer damped single degree of freedom. While this does not technically qualify as order reduction, it illustrates the basic methodology that will be used throughout this work. Then, we present the procedure to reduce a multiple integer-order degree of freedom system to a single fractional-order degree of freedom system. This represents the fundamental step of the order reduction methodology. Next, we extend the methodology to reduce a M-degree of freedom integer-order system to a N-degree of freedom fractional-order model where N < M. Lastly, we briefly discuss how the same methodology can be used in the frame of system identification and we show how to synthesize accurate fractional dynamic models based only on the knowledge of numerically obtained or experimentally measured data.

An important remark should be made concerning the terminology used in this paper. It is well-known that, for integer order systems, the overall order of the system is strictly related to the number of degrees of freedom because each individual degree is assumed to behave as a second order system. Therefore, order and dimension are typically considered as equivalent concepts and used interchangeably. On the contrary, due to the infinite dimensional character of a fractional derivative, the connection between the overall order of a system and the number of its physical degrees of freedom is somewhat more ambiguous. Such a discussion goes well beyond the scope of this paper and we simply highlight that, in the remainder, the term *degrees of freedom* will refer to the number of discrete masses while the term *order* will refer to the order of the individual differential equations.

2. Reduction to SDOF fractional systems

This section discusses the fundamental approach to obtain a fractional undamped single degree of freedom (SDOF) model exhibiting a dynamic behavior equivalent to an integer damped single degree of freedom system. To facilitate the understanding